Weighted Quadrature Domains and the Faber Transform

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Je n'ai fait celle-ci plus longue que parce que je n'ai pas eu le loisir de la faire plus courte. Blaise Pascal

ABSTRACT

This thesis develops the theory of weighted quadrature domains in parallel with the Faber transform as a tool for their analysis and explicit construction. Under this framework, we obtain a number of existence, uniqueness, and classification results for classical and weighted quadrature domains.

In the classical setting, we derive explicit formulae relating the Riemann map of a simply connected quadrature domain to its quadrature function via the Faber transform. This reduces the inverse and direct problems to a problem of solving finitely many algebraic equations. Applying these results, along with tools from logarithmic potential theory, we obtain a complete classification of one-point quadrature domains with complex charge.

We then introduce power-weighted quadrature domains (PQDs) - domains admitting a quadrature identity with respect to the weight $\rho_a(w) = |w|^{2(a-1)}$ for some $a > 0$. A central structural result is that a simply connected domain is a PQD if and only if the ath power of the outer factor of its Riemann map extends to a rational function. This characterization yields Faber transform formulae analogous to the classical case, which we apply to partially classify one-point and monomial PQD families. Novel boundary phenomena - including the formation of boundary "corners" at the origin with angles that are integer multiples of $\frac{\pi}{a}$ - are exhibited.

Next, we develop the theory of log-weighted quadrature domains (LQDs) - domains admitting a quadrature identity with respect to the weight $\rho_0(w) = |w|^{-2}$, the limiting case $a \to 0^+$ of ρ_a. The non-integrable singularity at the origin introduces new phenomena: when the domain contains the origin, the quadrature function is no longer uniquely determined, but only up to the addition of a point charge $\frac{q}{w}$. Despite this loss of uniqueness, we show that a simply connected domain is an LQD if and only if the outer factor of its Riemann map extends to the exponential of a rational function. Classification results for null, monomial, and one-point LQD families are obtained, and the connection to classical quadrature domains via the exponential map is established.

Finally, we introduce algebraic quadrature domains (AQDs), defined with respect to weights of the form $\rho_R = |R'|^2$, where R is a non-constant rational function. This class subsumes both classical quadrature domains ($R(w) = w$) and integer power-weighted quadrature domains ($R(w) = \frac{w^n}{n}$). We derive representation formulae in terms of the Faber transform and present several examples.

PUBLISHED CONTENT AND CONTRIBUTIONS

[1] Andrew Graven. *Analysis of Log-Weighted Quadrature Domains*. 2026. arXiv: 2604.10394 [math.CV]. URL: https://arxiv.org/abs/2604.10394.

[2] Andrew Graven and Nikolai G. Makarov. *Quadrature Domains and the Faber Transform*. 2025. arXiv: 2509.03777 [math.CV]. URL: https://arxiv.org/abs/2509.03777.

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NOMENCLATURE

Sets

$\mathbb{C}$	The complex plane.
$\widehat{\mathbb{C}}$	The Riemann sphere.
$\mathbb{D}$	The unit disk.
$\mathbb{D}_r(w_0)$	The disk of radius r centered at w_0.
$\mathbf{Cl}(\Omega)$	The closure of Ω.
$\mathbf{Int}(\Omega)$	The interior of Ω.
Ω^c	The complement of Ω.
$\partial\Omega$	The boundary of Ω.
Ω_*	The exterior of Ω: $\widehat{\mathbb{C}} \setminus \mathrm{Cl}(\Omega)$.

Function Spaces

$\mathcal{M}(\Omega)$	The space of meromorphic functions on Ω.
$H(\Omega)$	The space of analytic functions on Ω.
$H^\infty(\Omega)$	The Hardy space of bounded analytic functions on Ω.
$\mathcal{A}(\Omega)$	The space of analytic functions on Ω and continuous up to the boundary.
$\mathcal{A}_0(\Omega)$	The subspace of $\mathcal{A}(\Omega)$ consisting of functions which vanish at ∞.
$L^1_a(\Omega;\rho)$	The subspace of $\mathcal{A}(\Omega)$ consisting of functions which are L^1-bounded with respect to the weight ρ.
$\mathrm{Rat}(\Omega)$	The set of rational functions whose poles lie only in Ω.
$\mathrm{Rat}_0(\Omega)$	The subspace of $\mathrm{Rat}(\Omega)$ consisting of functions which vanish at ∞.

Functions and Operators

$f^\#$	Reflection of f about the unit circle: $f^\#(z) = \overline{f(\overline{z}^{-1})}$.
C_Ω	The Cauchy projection (2.3).
Φ_φ	The Faber Transform (2.3).
Φ_φ^{-1}	The Inverse Faber Transform (2.3).

F_n	The nth Faber polynomial: $F_n = \Phi_\varphi(z^n)$ (2.3).
W_n	The nth inverse Faber polynomial: $W_n = \Phi_\varphi^{-1}(w^n)$ (2.3).
φ	Riemann map.
ψ	φ^{-1}.
φ_{in}	The inner part of φ (2.4).
φ_{out}	The outer part of φ (2.4).
C^Ω	The Cauchy transform of Ω (3.1).
C_ρ^Ω	The $\rho-$weighted Cauchy transform of Ω (4.5).
S	The Schwarz function: $S \in \mathcal{M}(\Omega)$ extending continuously to the boundary such that $S(w) \doteq \overline{w}$ (3.1).
S_a	The generalized power-weighted Schwarz function: $S_a \in \mathcal{M}(\Omega)$ extending continuously to the boundary such that $S_a(w) \doteq \frac{1}{a}\overline{w}\lvert w\rvert^{2(a-1)}$ (4.2.1).
S_0	The generalized log-weighted Schwarz function: $S_0 \in \mathcal{M}(\Omega)$ extending continuously to the boundary such that $S_0(w) \doteq \frac{\ln\lvert w\rvert^2}{w}$ (5.1.4).
$S_{R,\Omega}$	The generalized algebraic Schwarz function: $S_{R,\Omega} \in \mathcal{M}(\Omega)$ extending continuously to the boundary such that $S_{R,\Omega}(w) \doteq R'(w)\overline{R(w)}$ (6.2.2).
b_λ	Blaschke factor: $b_\lambda(z) = \frac{\overline{\lambda}}{\lvert\lambda\rvert}\frac{z-\lambda}{\overline{\lambda}z-1}$.

Miscellaneous

dA	The normalized area measure: $dA = \frac{dxdy}{\pi}$.
$f \doteq g$	Denotes equality on the boundary: $f\vert_{\partial\Omega} = g\vert_{\partial\Omega}$, when the domain is unambiguous.
$\mathbf{QD}(h)$	The set of quadrature domains with quadrature function h (3.0.1).
$\mathbf{QD}_a(h)$	The set of power-weighted quadrature domains with quadrature function h (4.2.1).
$\mathbf{QD}_0(h)$	The set of log-weighted quadrature domains with quadrature function h (5.1.1).
$\mathbf{QD}_0(h;q)$	The set of $\Omega \in \mathrm{QD}_0(h)$ with charge q at 0 (5.1.2).
$\mathbf{QD}_R(h)$	The set of algebraic quadrature domains with quadrature function h (6.2.1).
$\mathbf{QD}_R(h;q_1,\ldots,q_j)$	The set of $\Omega \in \mathrm{QD}_R(h)$ with charges q_j at the poles p_j of R in Ω (6.2.2).

Chapter 1

INTRODUCTION AND SUMMARY OF RESULTS

1.1 Introduction

The purpose of this thesis is to develop the theory of *weighted quadrature domains*, and to demonstrate the utility of the Faber transform as a tool for analysis and explicit construction thereof. In particular, we develop explicit formulae relating the Riemann map of a simply connected quadrature domain to its quadrature function, and extend this framework to several classes of weighted quadrature domains. Applications to the existence, uniqueness, and classification of quadrature domains and their weighted generalizations are of central importance.

Quadrature domains. A *quadrature domain* is a plane domain $\Omega \subset \widehat{\mathbb{C}}$ such that the integrals of analytic functions on that domain against the area measure may be evaluated via a finite *quadrature identity*. The classical example of this is the *mean value property* for the disk: if f is analytic in the disk $\mathbb{D}_r(w_0)$, then

$$\int_{\mathbb{D}_r(w_0)} f\,dA = r^2 f(w_0), \tag{1.1}$$

where $dA = \frac{dxdy}{\pi}$ is the normalized area measure. The mean value property asserts that the disk is a quadrature domain whose quadrature identity consists of a single point evaluation. Far deeper, however, is the converse: disks are the *only* finitely connected bounded domains satisfying Equation 1.1 [1].

More generally, a bounded domain Ω is a quadrature domain if there exist finitely many nodes $a_1, \ldots, a_N \in \Omega$ with orders $m_1, \ldots, m_N$, and coefficients $c_{j,k} \in \mathbb{C}$ such that

$$\int_{\Omega} f\,dA = \sum_{j=1}^{N} \sum_{k=0}^{m_j} c_{j,k} f^{(k)}(a_j),$$

for every f in a suitable test class. This identity can be reformulated as a contour integral against a rational *quadrature function* h:

$$\int_{\Omega} f\,dA = \oint_{\partial\Omega} f(w)h(w)dw,$$

an observation that connects quadrature domains to the theory of the Schwarz function and boundary regularity [8, 30].

Context and motivation. Quadrature domains appear across complex analysis, potential theory, and mathematical physics. For example,

1. *Hele-Shaw flow.* The complement K of a quadrature domain Ω is a "droplet" - the support of the equilibrium measure associated to a Hele-Shaw potential (Lemma 2.2.1). Under steady injection of fluid into the droplet, the boundary evolves via Laplacian growth, generating a one-parameter family of droplets. Hence, we obtain a corresponding one-parameter family of quadrature domains. This perspective provides a dynamical interpretation of families of quadrature domains and was developed systematically by Richardson [27], Lee-Makarov [20], and Hedenmalm-Makarov [16, 15].

2. *Potential theory and the obstacle problem.* Quadrature domains are dual to solutions of the classical obstacle problem in logarithmic potential theory [28]. Given an admissible external field Q, the equilibrium measure is supported on a compact droplet whose complement is a disjoint union of quadrature domains. Conversely, recovering the quadrature domain from its quadrature function is equivalent to solving the obstacle problem for the associated potential. This duality, formalized in Lemma 2.2.1, underlies several important existence and uniqueness arguments in this thesis.

3. *Random matrix theory.* The eigenvalue distribution of certain ensembles of random normal matrices converges, in the large N limit, to the equilibrium measure of a logarithmic potential. The boundary of the support of the limiting measure is the boundary of a quadrature domain. See [24] for a survey of these connections.

4. *The Schwarz function and conformal dynamics.* The set of quadrature domains consists precisely of those domains which admit a *Schwarz function* S (Proposition 3.1.1). Furthermore, the associated quadrature identity can be directly obtained from the poles of this function [8]. Iteration of the associated Schwarz reflection $\sigma := \overline{S}$ (which is the identity on the boundary) yields a complex dynamical system with important topological [21] and algebraic implications. For example, these topological bounds imply bounds on the number of solutions to the equation
$$r(z) = \overline{z},$$
where r is a rational function. These directly imply bounds on image duplication in gravitational lensing [5].

Several specific classes of quadrature domains have been known since at least 1907, when Neumann observed that certain ovals satisfy higher-order mean value properties [25]. However, the systematic study of quadrature domains did not begin until the early 1970s, with the foundational work of Richardson [27], Davis [8], and Aharonov & Shapiro [1], who introduced the general definition and established major structural theorems. Since then, the subject has developed considerably; we refer to [9, 14, 20, 30] for further background.

The inverse and direct problems. Two fundamental questions organize much of the theory:

- **The inverse problem:** Given a quadrature function h, reconstruct the domain Ω.
- **The direct problem:** Given a domain Ω, determine its quadrature function h.

For simply connected domains, both problems reduce to determining the relationship between two collections of rational data: the quadrature function h and the Riemann map φ. Making this relationship explicit and computable is a central goal of this thesis.

The Faber transform. The main tool we employ is the *Faber transform*, a linear operator between analytic functions on the disk $\mathcal{A}(\mathbb{D})$ and analytic functions on a given simply connected domain $\mathcal{A}(\Omega)$ (see §2.3 for the formal definition). For our purposes, the Faber transform has three important properties:

1. It is an *isomorphism*: every function analytic in Ω_* (the exterior of Ω) is the Faber transform of a unique function analytic in the disk and vice versa.
2. It *preserves rational functions*: the Faber transform of a rational function is rational.
3. It is *effectively computable*: the Faber transform of a rational function can be evaluated in finitely many steps via calculus of residues (Equation 2.9).

The Faber transform and the associated Faber polynomials have been studied extensively in approximation theory [10, 3, 18]; Chang [6] observed that the Faber transform provides a natural framework for solving the inverse problem for bounded quadrature domains. In this thesis, building on the approach initiated in [13], we systematically develop the Faber transform as a tool for solving both the inverse and direct problems, for bounded and unbounded domains, and for several classes of weighted quadrature domains.

The key formulae take the following form: If Ω is a bounded simply connected quadrature domain with Riemann map $\varphi : \mathbb{D} \to \Omega$ and quadrature function h, then:

$$h = \Phi_\varphi \left(C_{\mathbb{D}_*} \left[\varphi^\# \right] \right), \quad \varphi = \varphi(0) + \Phi_\varphi^{-1} (h)^\#,$$

where Φ_φ denotes the Faber transform and $f^\#(z) := \overline{f(\overline{z}^{-1})}$ is the reflection about the unit circle (Theorem 3.2.2). Analogous formulae hold in the unbounded case (Theorem 3.2.3). Because both h and the relevant data of φ are rational, these formulae reduce the inverse and direct problems to one of solving finitely many algebraic equations in the coefficients of φ and h.

Weighted quadrature domains. A substantial portion of this thesis is devoted to extending the classical theory to *weighted* quadrature identities, in which the area measure dA is replaced by ρdA for a suitable weight function ρ. We consider three progressively more general classes of weights.

Power-weighted quadrature domains (§4) are defined by the weight $\rho_a(w) = |w|^{2(a-1)}$ $(a > 0)$. This choice is motivated by the correspondence between power-weighted quadrature domains and local droplets of potentials of the form

$$Q(w) = \frac{|w|^{2a}}{a^2} - 2\mathrm{Re}(H(w)),$$

where $H' = h$ is the quadrature function. A key structural result (Theorem 4.3.3) is that a simply connected domain is a power-weighted quadrature domain if and only if the ath power of the outer factor of its Riemann map extends to a rational function. This characterization leads to Faber transform formulae analogous to the classical case and enables the classification of several families of power-weighted quadrature domains, including one-point and monomial families ($h(w) = \frac{\alpha}{w-w_0}$ and $h(w) = \alpha k w^{k-1}$ respectively).

Log-weighted quadrature domains (§5) are characterized by the weight $\rho_0(w) = |w|^{-2}$, which corresponds to the limit as $a \to 0^+$ of ρ_a. Unlike the power weights, ρ_0 has a non-integrable singularity at the origin, which introduces fundamentally new phenomena. When $0 \in \Omega$, admissible test functions all vanish at the origin, and as a consequence the quadrature function is no longer uniquely determined by the domain but only up to the addition of a point charge q/w at the origin. Despite this loss of uniqueness, we show that the simply connected theory still admits a clean formulation: a domain is a log-weighted quadrature domain if and only if the outer factor of its Riemann map extends to the exponential of a rational function (Theorem 5.4.1). The corresponding Faber transform formulae are derived, and classification results for null, monomial, and one-point log-weighted quadrature domains are obtained. Log-weighted quadrature domains arise naturally as images of classical quadrature domains under the exponential map, and they appear as complements of local droplets of the logarithmic potential ([12], §4.1.4):

$$Q(w) = \frac{1}{2}\ln^2|w|^2 - 2\mathrm{Re}(H(w)).$$

Algebraic quadrature domains (§6) are defined with respect to weights of the form $\rho_R = |R'|^2$, where R is a non-constant rational function. This class subsumes both the classical case ($R(w) = w$) and the integer power-weighted case ($R(w) = \frac{w^n}{n}$, giving $\rho_R(w) = |w|^{2(n-1)} = \rho_n(w)$). The identity $|R'|^2 = \frac{1}{4}\Delta|R|^2$ ensures that the generalized Schwarz function machinery applies, and a change-of-variables principle (§6.3) reduces many questions about algebraic quadrature domains to their classical counterparts via the change of coordinates $w \mapsto R(w)$.

Relationship to published work. Chapters 3 and 4, together with portions of Chapter 5, are based on the joint paper with N.G. Makarov [13]. There, we develop the Faber transform method for classical and power-weighted quadrature domains, obtain a complete classification of one-point quadrature domains with complex charge (§3.3), and introduce the theory of log-weighted quadrature domains in the non-singular setting. The extension to singular log-weighted quadrature domains - those containing the origin - is the subject of a separate paper by the author [12], which provides the material for the singular portions of Chapter 5, as well as a more natural treatment of the weighted Faber transform identities.

1.2 Notation

General notation: Throughout, $\widehat{\mathbb{C}}$ denotes the Riemann sphere. A domain is a connected open subset of $\widehat{\mathbb{C}}$. For a set $\Omega \subseteq \widehat{\mathbb{C}}$, we denote by $\mathrm{Cl}(\Omega)$ its closure, Ω^c its complement, $\partial\Omega$ its boundary, $\mathrm{Int}(\Omega)$ its interior, and

$$\Omega_* := \widehat{\mathbb{C}} \setminus \mathrm{Cl}(\Omega)$$

the exterior of Ω. We call a domain rectifiable if $\partial\Omega$ is rectifiable. When f and g are functions defined on $\partial\Omega$, we write $f \doteq g$ to denote equality on $\partial\Omega$. $f^{\#}(z) := \overline{f(\overline{z}^{-1})}$ denotes the reflection of f about the unit circle.

$\ln(w)$ denotes the principal branch of the natural logarithm (branch cut along the negative real axis), unless otherwise stated.

Contour integrals are normalized such that the standard factor of $2\pi i$ is suppressed. $dA = \frac{dxdy}{\pi}$ denotes the normalized area measure. If Ω is an unbounded domain, then integrals are understood in terms of their Cauchy principal value,

$$\int_\Omega f\,dA := \lim_{R\to\infty} \int_{\Omega\cap\mathbb{D}_R} f\,dA.$$

Function spaces: If Ω is a domain, we denote by $\mathcal{M}(\Omega)$ the space of meromorphic functions on Ω, $H(\Omega)$ the space of analytic functions on Ω, and $H^\infty(\Omega)$ the Hardy space of bounded analytic functions on Ω. We define:

$$A(\Omega) := H(\Omega) \cap C^0(\mathrm{Cl}(\Omega)),$$

the space of functions analytic in Ω which extend continuously to the boundary.

If Ω is unbounded, we define:

$$\mathcal{A}_0(\Omega) := \{f \in \mathcal{A}(\Omega) : f(\infty) = 0\},$$

the $\mathcal{A}(\Omega)$-subspace of functions which vanish at ∞.

For a non-negative weight ρ, we denote by $L^1_a(\Omega;\rho)$ the subspace of $\mathcal{A}(\Omega)$ consisting of functions f which are L^1−bounded with respect to the weight ρ:

$$\int_\Omega |f|\rho\,dA < \infty.$$

We denote by $\mathrm{Rat}(\Omega)$ the set of rational functions whose poles lie only in Ω, and $\mathrm{Rat}_0(\Omega)$ the subclass vanishing at ∞.

1.3 Overview and Main Results

We now give a chapter-by-chapter overview of the main results of this thesis.

Chapter 2: Background. This chapter collects the prerequisites from potential theory, Hele-Shaw flow, and classical function theory that are used throughout the thesis.

The first key ingredient is the *obstacle problem* from logarithmic potential theory. Given an admissible external field Q, Frostman's theorem (Proposition 2.1.1) asserts the existence and uniqueness of a compactly supported equilibrium measure $\widehat{\mu}$, and the support $K_{\widehat{\mu}}$ of this measure is called a *droplet*. The Gauss variational problem (Proposition 2.1.2) refines this by showing that when Q is sufficiently smooth, the equilibrium measure has density $\frac{\Delta Q}{2}\, dA$ on its support, and the support is determined by a *coincidence equation* (Equation 3.3).

The second key ingredient is the duality between quadrature domains and droplets, established via Hele-Shaw flow. Lemma 2.2.1 asserts that the complement of a local droplet of a Hele-Shaw potential $Q(w) = |w|^2 - 2\mathrm{Re}(H(w))$ is a disjoint union of quadrature domains, and conversely. This duality provides a dynamical interpretation of quadrature domain families: as a backward Hele-Shaw chain contracts, the complementary QDs expand in a monotone family parameterized by area (Lemma 2.2.2).

The third key ingredient is the *Faber transform* (Definitions 2.3.1 and 2.3.2), a linear isomorphism between analytic functions on the disk and analytic functions on a simply connected domain Ω. Its defining property for our purposes is that it takes rational functions to rational functions, and it can be computed explicitly via the calculus of residues (Equation 2.9), with closed-form representations in terms of the incomplete Bell polynomials (Equation 2.10).

Finally, we review the *inner/outer factorization* of Riemann maps (Theorems 2.4.1 and 2.4.2), which decomposes the Riemann map φ into an inner function φ_{in} (given by a ratio of Blaschke factors) and an outer factor φ_{out} given by a Poisson-type integral. This factorization can be rewritten in terms of the Faber transform (Theorem 2.14), and plays a central role in the construction of weighted quadrature domains in Chapters 4 and 5.

Chapter 3: Quadrature Domains. This chapter develops the theory of classical quadrature domains (QDs). That is, domains satisfying an identity of the form

$$\int_{\Omega} f\, dA = \oint_{\partial\Omega} f(w)h(w)dw,$$

for functions analytic in Ω for which both integrals converge, and h is a rational function (the *quadrature function*). This is denoted by $\Omega \in \mathrm{QD}(h)$. We establish a method of reconstructing

QDs from their quadrature function via the Faber transform, and apply this method to fully classify general one-point QDs.

After reviewing the basic definitions and properties of QDs - the Schwarz function characterization (Proposition 3.1.1), the Sakai regularity theorem (Theorem 3.1.1), and the classical result that a simply connected domain is a QD if and only if it has a rational Riemann map (Theorem 3.2.1) - we turn to the central Faber transform formulae of the chapter.

Theorem (3.2.2)**.** *Let $\Omega \in QD(h)$ be a bounded and simply connected domain with quadrature function $h \in \mathrm{Rat}_0(\Omega)$ and Riemann map $\varphi : \mathbb{D} \to \Omega$. In this case,*

$$h = \Phi_\varphi \left(\varphi^\# - \overline{\varphi(0)} \right)$$

and

$$\varphi = \varphi(0) + \Phi_\varphi^{-1}(h)^\#.$$

Theorem (3.2.3)**.** *Let $\Omega \in QD(h)$ be an unbounded and simply connected domain with quadrature function $h \in \mathrm{Rat}(\Omega)$ and Riemann map $\varphi : \mathbb{D}_* \to \Omega$. In this case,*

$$h = \Phi_\varphi \left(\varphi^\#(z) - cz^{-1} \right)$$

and

$$\varphi(z) = cz + \Phi_\varphi^{-1}(h)^\#(z).$$

Where $c = \varphi'(\infty)$ is the conformal radius of Ω and Φ_φ is the Faber transform (§2.3). These theorems solve the direct and inverse problems for simply connected bounded and unbounded QDs respectively. Because both h and φ are rational, the formulae reduce these problems to problems of solving finitely many algebraic equations.

We apply these formulae to obtain a complete classification of one-point quadrature domains, including those with *complex charge* α. The case of positive real charge ($\alpha > 0$) has been studied extensively (e.g. [9]), but the complex case had received little attention. The following theorem summarizes this classification:

Theorem (3.3.1)**.** *Take $\alpha \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$ and $w_0 \neq 0$. Then*

$$QD\left(\frac{\alpha}{w - w_0}\right) \neq \emptyset \qquad \Longleftrightarrow \qquad |w_0|^2 + 2Re\,(\alpha) > 2|\alpha|.$$

In this case, there exists $t_* \in (0, \infty)$ such that

1. *$QD\left(\frac{\alpha}{w-w_0}\right) = \{\Omega_t\}_{0<t\leq t_*}$, a continuous monotone (2.2.2) family of simply connected domains (where $t = A(\Omega_t)$);*

2. *$\partial\Omega_t$ is smooth for all $0 < t < t_*$ and the family terminates with the formation of a $(3,2)$ cusp at $t = t_*$.*

The existence criterion describes a parabolic region in the α-plane, and the cusp formation at the critical time generalizes the double-point formation that occurs when $\alpha > 0$. The Riemann map of each Ω_t is given explicitly by Equation 3.10 (Corollary 3.3.1). Several families of solutions are illustrated in Figures 3.1, 3.2, and 3.3.

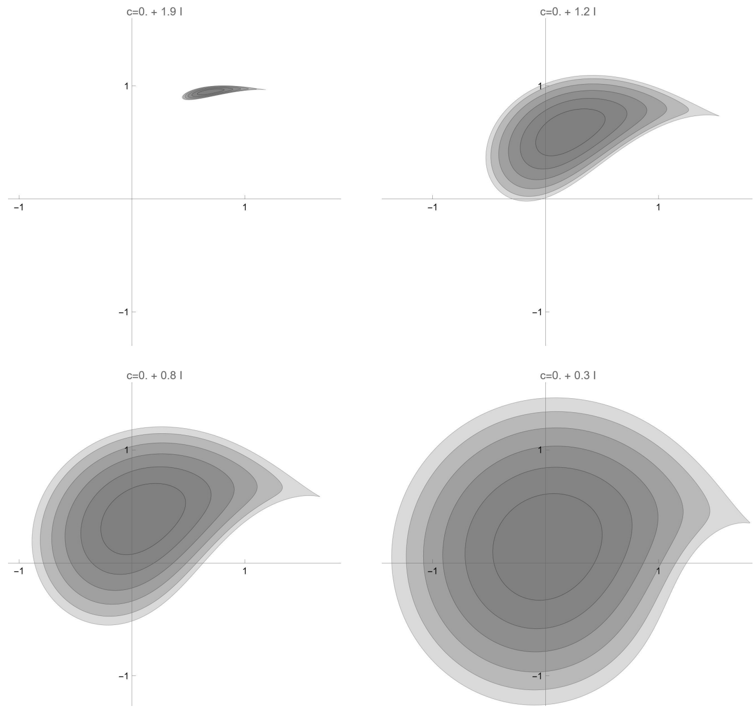

Figure 3.3: One-point quadrature domains $\Omega \in \mathrm{QD}\left(\frac{\alpha}{w-2}\right)$ for $\alpha \in \{1.9i, 1.2i, .8i, .3i\}$ (complements of shaded regions) for conformal radius increasing to its maximal value, at which point a $(3,2)$-cusp forms on the boundary.

Chapter 4: Power-Weighted Quadrature Domains. This chapter introduces and develops the theory of *power-weighted quadrature domains* (PQDs): domains Ω admitting a quadrature identity with respect to the weight $\rho_a(w) = |w|^{2(a-1)}$ for some $a > 0$:

$$\int_\Omega f\rho_a dA = \oint_{\partial\Omega} f(w)h(w)dw,$$

for all f analytic in Ω for which both integrals converge, and h a rational function. This is denoted by $\Omega \in \mathrm{QD}_a(h)$. We begin more broadly with the notion of weighted quadrature domains (Definition 4.1.1) and establish a generalized coincidence equation (Proposition 4.1.1) for weights of the form $\rho = \frac{1}{4}\Delta F$ with F radially symmetric. Specializing to $F(w) = \frac{|w|^{2a}}{a^2}$ yields the generalized coincidence equation for PQDs (Equation 4.3), together with an equivalent characterization in terms of the *generalized Schwarz function* (Theorem 4.2.1).

A key difference from the classical theory is that the weight ρ_a is singular at the origin when $a \neq 1$ ($\rho_a(0) = 0$ when $a > 1$ and $\rho_a(0) = \infty$ when $0 < a < 1$), and this singular point gives rise to boundary phenomena excluded in the classical case by Sakai regularity: boundaries of PQDs can develop *corners* at the origin, with angles that are integer multiples of $\frac{\pi}{a}$ (see Figures 4.2, 4.6, 4.7). We prove a regularity result for PQDs (Theorem 4.2.3): if $\Omega \in \mathrm{QD}_a$ for some $a \in \mathbb{Z}_+$, then the boundary $\partial\Omega$ has finitely many singular points and each non-zero singular point is either a cusp or a double point. We furthermore show that $(\partial\Omega)^a$ is an algebraic curve when $a \in \mathbb{Z}_+$ (Theorem 4.2.2).

The central structural result for simply connected PQDs is the following characterization in terms of the inner/outer factorization of the Riemann map. Let Ω be a simply connected domain with Riemann map $\varphi = \varphi_{\text{in}}\varphi_{\text{out}}$. Then,

Theorem (4.3.3)**.** *$\Omega \in QD_a$ if and only if φ_{out}^a extends to a rational function.*

This characterization leads to explicit Faber transform formulae solving the inverse and direct problems for PQDs (Theorems 4.3.1 and 4.3.2 for the sum identities; Theorems 4.3.4 and 4.3.6 for the “inner/outer” factorization identities).

These general results are then applied to several concrete families. In §4.4, we classify simply connected one-point PQDs (Theorems 4.4.2 and 4.4.3). In §4.5, we classify monomial PQDs - those with quadrature function $h(w) = \alpha k w^{k-1}$ for some $k \in \mathbb{Z}_{\geq 0}$ and $\alpha \in \mathbb{C}$ (Theorems 4.5.1 and 4.5.3). In §4.6, we present further examples including linear, quadratic, and binomial quadrature functions, illustrating the rich variety of boundary phenomena (cusps, corners, double points) that arise in the power-weighted setting.

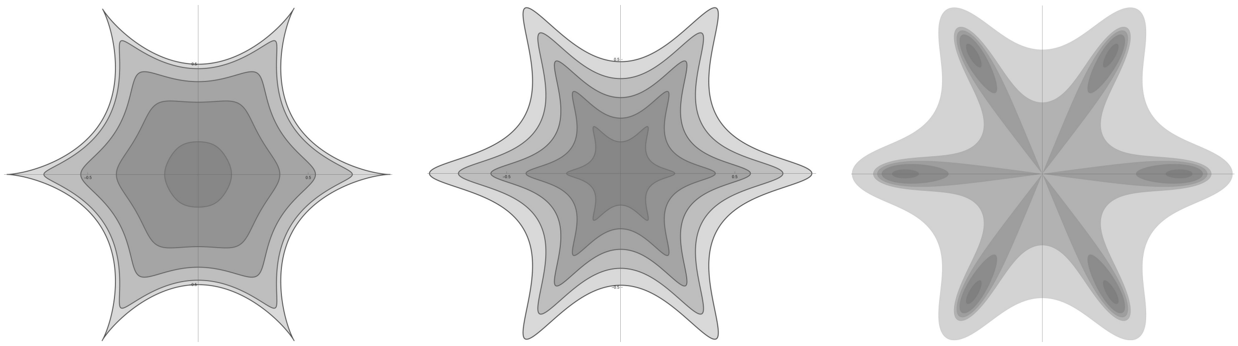

Figure 4.6: Families of monomial PQDs (complements of shaded regions) in $\mathrm{QD}_{a=2}\left(\frac{w^{k-1}}{2}\right)$, $\mathrm{QD}_{a=3}\left(\frac{2}{9}w^{k-1}\right)$, and $\mathrm{QD}_{a=4}\left(\frac{w^{k-1}}{8}\right)$ with $k = 6$ (left to right). When $a = 2$, $k > 2a$, so the family has a maximal element. When $a = 3$, $k = 2a$, so this is the "traveling wave" case. Finally, when $a = 4$, $k < 2a$, so we obtain a two-phase family which is initially multiply connected, transitioning to simply connected past the critical time.

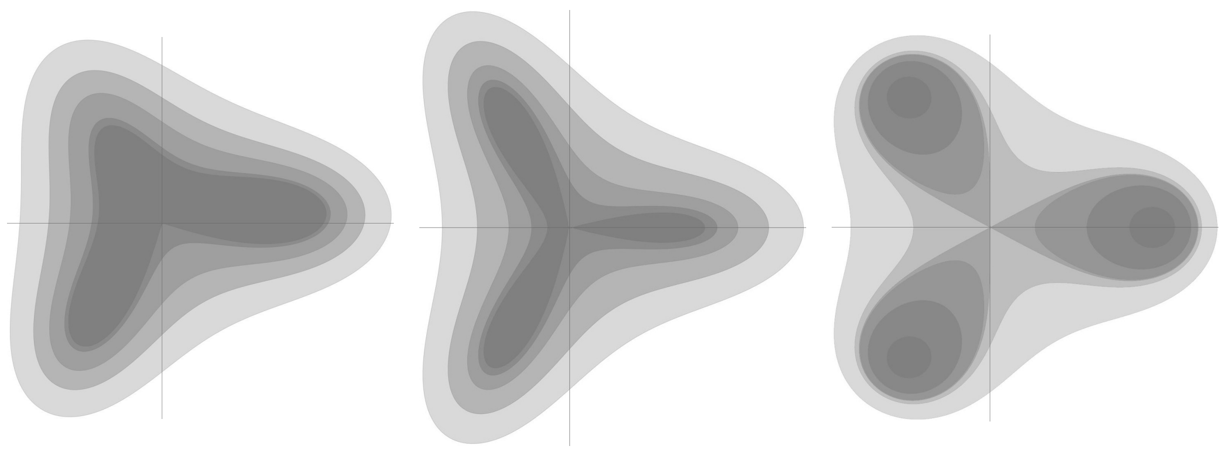

Figure 4.10: Families of unbounded PQDs (complements of shaded regions) satisfying the weighted quadrature identity:

$$\int_\Omega f(w)4|w|^2 dA(w) = \oint_{\partial\Omega} f(w)(\alpha_0 + \alpha_1 w + w^2)dw,$$

for all $f \in \mathcal{A}_0(\Omega)$, for various values of $\alpha_1, \alpha_2 \in \mathbb{C}$. Each figure shows a family of domains with the same quadrature function, parameterized by conformal radius.

Chapter 5: Log-Weighted Quadrature Domains. This chapter develops the theory of *log-weighted quadrature domains* (LQDs): domains satisfying a quadrature identity with respect to the weight $\rho_0(w) = |w|^{-2}$:

$$\int_\Omega f\rho_0 dA = \oint_{\partial\Omega} f(w)h(w)dw,$$

for all $f \in L^1_a(\Omega; \rho_0)$, and h a rational function. This is denoted by $\Omega \in \mathrm{QD}_0(h)$. LQDs ($\mathrm{QD}_0$) can be understood as a limit of PQDs (QD_a) as $a \to 0^+$ (Theorem 5.2.1), and they arise as images of classical QDs under the exponential map.

Theorem (5.1.1)**.** *If $\Omega \in QD(h)$ is a bounded domain on which $w \mapsto e^w$ is injective, then $e^\Omega \in QD_0(\widetilde{h})$, where $\widetilde{h}(w) = C_{e^\Omega_*}\left[\frac{h\circ\ln(w)}{w}\right]$.*

A key new phenomenon arises in the *singular* case, when $0 \in \Omega$. In the classical and non-singular theories, the quadrature function h is uniquely determined by the domain (when h is restricted to the appropriate class). However, when $0 \in \Omega$, every admissible test function vanishes at the origin, and as a result the quadrature function is determined only up to the addition of a point charge $\frac{q}{w}$ at the origin (Theorem 5.1.3). This requires new formulations of the coincidence equation, the Schwarz function, and the direct/inverse problems.

In the non-singular case, the coincidence equation takes the form (Theorem 5.1.2)

$$\frac{\ln|w|^2}{w} \doteq h(w) + G(w), \quad G \in \mathcal{A}(\Omega).$$

In the singular case, it becomes (Theorem 5.1.3)

$$\frac{\ln|w|^2}{w} \doteq h(w) + \frac{q}{w} + G(w), \quad q \in \mathbb{C}, \; G \in \mathcal{A}(\Omega).$$

From these, we derive a generalized Schwarz function characterization (Theorem 5.1.4), boundary regularity results (Theorem 5.1.5), and an electrostatic interpretation (Theorem 5.1.6): an LQD can be understood as a domain such that the electrostatic field due to ρ_0-weighted charge density on Ω can be replicated on the exterior of Ω by the placement of finitely many point charges (and multipoles) on the interior of Ω.

The central result for simply connected LQDs parallels our Riemann map characterization of PQDs (Theorem 4.3.3):

Theorem (5.4.1)**.** $\Omega \in QD_0$ *if and only if φ_{out} extends to the exponential of a rational function.*

As with PQDs, this characterization yields explicit Faber transform formulae relating r (and hence φ) to the quadrature function h (Theorem 5.4.2).

We then apply these results to classify several families of LQDs. We give a complete classification of *null LQDs* - LQDs whose quadrature function is $h = 0$:

Theorem (5.3.1)**.** *A domain (not necessarily simply connected) is a null LQD if and only if it is a disk or exterior disk centered at the origin.*

We furthermore characterize simply connected non-singular one-point LQDs (e.g. Figure 5.5, top row):

Theorem (5.3.2)**.** *Fix $\alpha, w_0 \in \mathbb{C} \setminus \{0\}$. If Ω is a simply connected bounded domain not containing zero, then $\Omega \in QD_0\left(\frac{\alpha}{w-w_0}\right)$ if and only if $0 < \alpha \leq \pi^2$ and $\Omega = \{w \in \mathbb{C} : |\ln(ww_0^{-1})|^2 < \alpha\}$.*

On the other hand, for simply connected non-singular unbounded one-point LQDs, the Riemann map takes the form

$$\varphi(z) = cze^{\frac{\lambda}{1-z\overline{z_1}}}$$

(Theorem 5.6.1), and we give a complete characterization of the parameters for which φ is univalent. Hence, we obtain a complete characterization of the family of unbounded one-point LQDs not containing zero. The classification of monomial LQDs is addressed in §5.5.

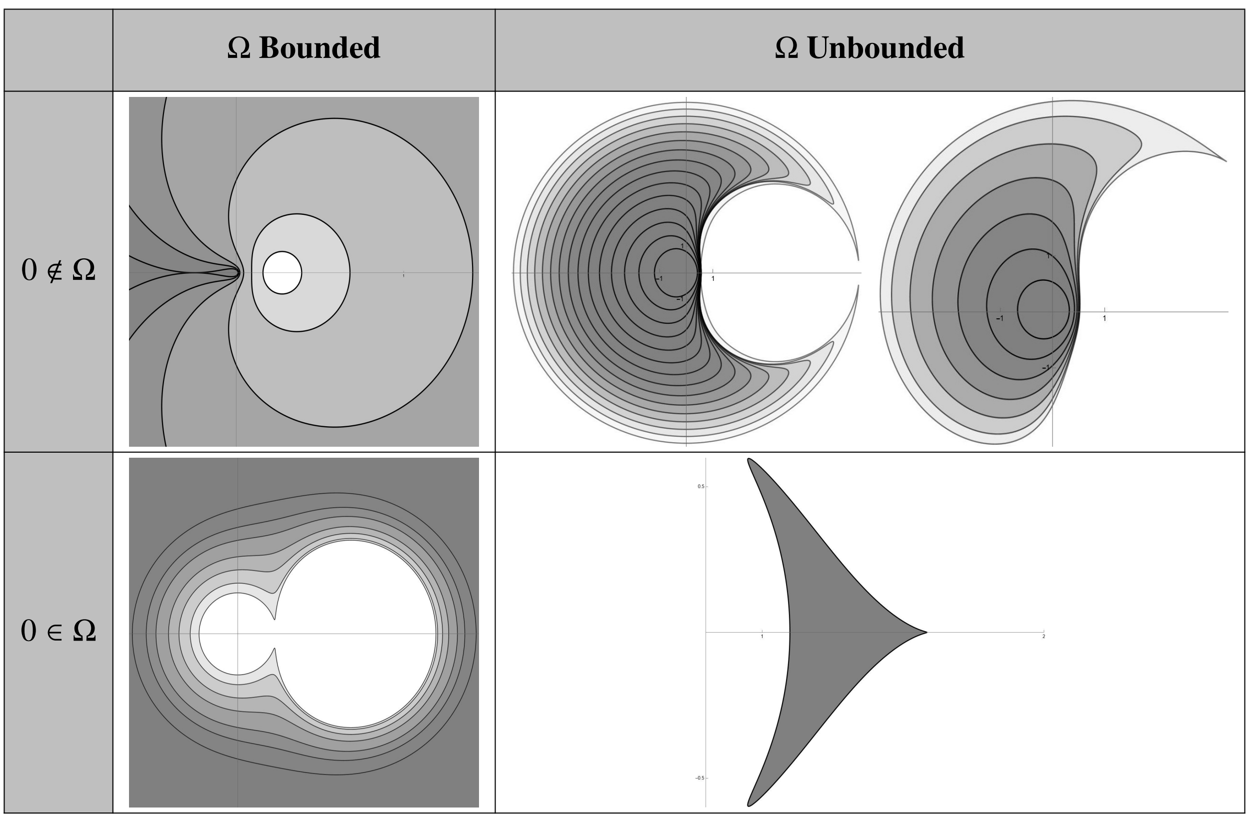

Figure 5.5: Families of LQDs (complements of shaded regions) satisfying the weighted quadrature identity:

$$\int_\Omega \frac{f(w)}{|w|^2} dA(w) = \alpha f(w_0),$$

for all $f \in L^1_a(\Omega; \rho_0)$, for various values of $w_0 \neq 0$ and $\alpha \in \mathbb{C}$.

Chapter 6: Algebraic Quadrature Domains. This chapter broadens the scope to domains admitting a quadrature identity with respect to an *algebraic weight* of the form $\rho_R = |R'|^2$:

$$\int_{\Omega} f\rho_R dA = \oint_{\partial\Omega} f(w)h(w)dw,$$

for all f analytic in Ω for which both integrals converge, and h, R are rational functions. This is denoted by $\Omega \in \mathrm{QD}_R(h)$. We refer to such domains as *algebraic quadrature domains* (AQDs). This class is natural for several reasons: the identity $|R'|^2 = \frac{1}{4}\Delta|R|^2$ ensures that the generalized Schwarz function machinery (Proposition 4.1.1) applies; the case $R(w) = w$ recovers classical QDs; and the case $R(w) = \frac{w^a}{a}$, $a \in \mathbb{Z}_+$, recovers PQDs.

An important structural tool in this chapter is a *change-of-variables principle*:

Proposition (6.3.1)**.** Let $\Omega, \Omega_R \subset \widehat{\mathbb{C}}$ be domains and suppose $R : \Omega_R \to \Omega$ is a rational diffeomorphism with $R \in C^0(\partial\Omega_R)$. Then $\Omega \in \mathrm{QD}(h)$ if and only if $\Omega_R \in \mathrm{QD}_R(h_R)$, where h and h_R are related by

$$h_R(w) = C_{(\Omega_R)_*}\left[h \circ R \cdot R'\right](w) = \oint_{\partial\Omega_*} \frac{h(\xi)d\xi}{R^{-1}(\xi) - w},$$
$$h(w) = C_{\Omega_*}\left[\left(\frac{h_R}{R'}\right) \circ R^{-1}\right](w) = \oint_{\partial(\Omega_R)_*} \frac{h_R(\xi)d\xi}{R(\xi) - w}.$$

This principle reduces many questions about AQDs to their classical counterparts via the map $\mathrm{QD}_R \xrightarrow{R} \mathrm{QD}$ (which only makes sense for domains on which R is injective).

In the simply connected case, we develop Faber transform formulae for AQDs that generalize the formulae for classical QDs (Theorems 3.2.2 and 3.2.3) and for PQDs (Theorems 4.3.2 and 4.3.1):

Theorem (6.4.1)**.** *Suppose $\Omega \in QD_R(h)$ is a bounded simply connected domain with Riemann map $\varphi : \mathbb{D} \to \Omega$, where $\varphi(0) = w_0$ is not a pole of R. In this case,*

$$R \circ \varphi(z) = R(w_0) + \Phi_\varphi^{-1}\left(C_{\Omega_*}[R]\right)(z) - \Phi_\varphi^{-1}\left(C_{\Omega_*}[R]\right)(0) + r^{\#}(z),$$

where r is a rational function determined explicitly by the quadrature function (Equation 6.11).

An analogous result holds in the unbounded case (Theorem 6.4.2). We furthermore prove the following generalization of the characterization of simply connected quadrature domains as those whose Riemann map is rational (Theorem 3.2.1):

Theorem (6.4.4)**.** *Let R be rational and Ω a simply connected domain with Riemann map φ and $\partial\Omega$ containing no poles of R. Then $\Omega \in QD_R$ if and only if $R \circ \varphi$ extends to a rational function.*

We conclude Chapter 6 by presenting several explicit examples of AQDs, including a generalization of PQDs to negative values of a and AQDs containing multiple singularities of R.

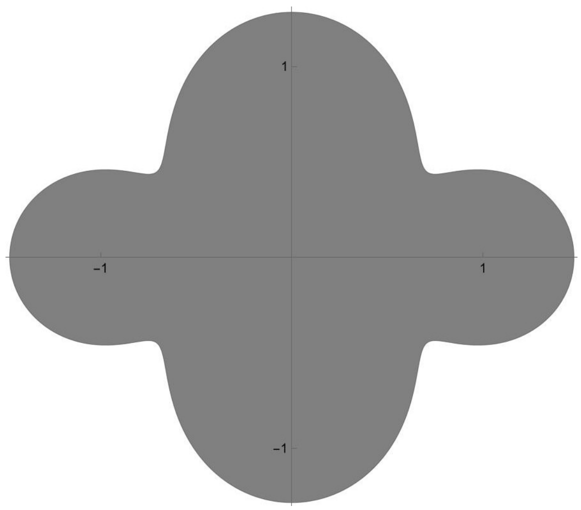

Figure 6.2: An AQD satisfying a weighted quadrature identity with singularities at $w = \pm 1$:

$$\int_\Omega f(w) \frac{|w|^2}{|w^2-1|^4} dA(w) = \frac{f(0)}{6},$$

for all $f \in \mathcal{A}(\Omega)$ with $f(\pm 1) = 0$, so the integral converges in $R-$principal value (6.1).

Chapter 2

BACKGROUND

This chapter collects the background material used throughout the thesis. The four ingredients serve complementary roles. Logarithmic potential theory (§2.1) supplies a geometric framework: the equilibrium measure and its support - the droplet - provide existence and uniqueness criteria for quadrature domains via the obstacle problem. Hele-Shaw flow (§2.2) connects to the dynamical nature of QDs, establishing a duality between potential-theoretic droplets and quadrature domains that organizes QDs into monotone one-parameter families and governs the formation of boundary singularities. The Faber transform (§2.3) is the principal computational tool: as a linear isomorphism of analytic function spaces which preserves rational functions and admits formulae enabling explicit computation, it reduces the inverse and direct problems for simply connected QDs to finite collections of algebraic equations in the coefficients of the Riemann map and the quadrature function. Finally, the inner/outer factorization of the Riemann map (§2.4) becomes essential in the weighted setting of Chapters 4 and 5, where the condition for a domain to be a weighted QD is captured by whether the outer factor φ_{out} of the Riemann map φ may be expressed in terms of rational functions.

2.1 Potential Theory

Let μ be a non-negative measure on $\mathbb{C}$ with compact support, $K_\mu = \operatorname{supp}(\mu)$. The logarithmic potential and logarithmic energy associated to μ are respectively given by

$$U^\mu(w) = \int_{\mathbb{C}} \ln \frac{1}{|w-\xi|} d\mu(\xi), \qquad I(\mu) = \int_{\mathbb{C}} U^\mu d\mu.$$

Now suppose that we have an admissible external field Q. By *admissible*, we mean $Q : \mathbb{C} \to (-\infty, \infty]$ is lower-semicontinuous and satisfies the growth condition

$$\lim_{|w|\to\infty} (Q(w) - \ln|w|) = \infty.$$

In this case, the $Q-$potential and $Q-$energy (also referred to as the weighted logarithmic capacity of μ) are respectively given by

$$U^\mu_Q(w) = \int_{\mathbb{C}} \left(\ln \frac{1}{|w-\xi|} + Q(\xi) + Q(w)\right) d\mu(\xi), \qquad I_Q(\mu) = \int_{\mathbb{C}} U^\mu_Q d\mu.$$

If Q is admissible, then there exists a unique $Q-$energy minimizing measure μ, $\mu(1) = 1$, called the *equilibrium measure* associated to Q. This is stated rigorously in the following theorem due to Frostman [11].

Proposition 2.1.1 (Frostman)**.** If an external field Q is admissible then there exists a measure $\widehat{\mu}$ such that

$$I_Q(\widehat{\mu}) = \gamma := \inf_{\mu:\mu(1)=1} I_Q(\mu).$$

Moreover, $\widehat{\mu}$ is the unique compactly supported measure for which $U_Q^{\widehat{\mu}} = \gamma$ on $K_{\widehat{\mu}}$ and $U_Q^{\widehat{\mu}} \geq \gamma$ q.e. (up to a set of capacity zero) in $\mathbb{C}$.[1] We refer to the support of the equilibrium measure $K_{\widehat{\mu}}$ as a "droplet".

Note that even if Q fails to satisfy the desired growth condition at ∞, we can still consider the obstacle problem *locally*. In particular, if X is a compact subset of $\mathbb{C}$, then we can localize to X by instead considering the obstacle problem associated to the potential $Q_X(w) = Q(w)\mathbb{1}_X + \infty\mathbb{1}_{X^c}$. We refer to the support of the equilibrium measure in this case as a *local droplet*.

Now suppose that Q satisfies the stronger growth condition,

$$\lim_{|w|\to\infty} (Q(w) - t\ln|w|) = \infty, \;\; \forall t \in \mathbb{R}.$$

By Frostman's theorem, there exists a unique equilibrium measure $\widehat{\mu}$ associated to Q. The Gauss variational (or "obstacle") problem is to find

$$v_t(w) = \sup\{v(w) : v \leq Q \text{ q.e., } v \text{ is subharmonic in } \mathbb{C}, \text{ and } \sim t\ln|w|\}.$$

A direct consequence of this definition is that $v_t(w) = U_Q^{\mu_t}(w) + c(t)$, where $\mu_t = t\widehat{\mu}$, for $\widehat{\mu}$ the equilibrium measure associated to $t^{-1}Q$, and $c(t)$ is a constant. Furthermore if $Q \in W^{2,p}, \;\; p > 1$ in an open neighborhood of $K_t := \text{supp}(\mu_t)$, then

$$d\mu_t = \mathbb{1}_{K_t}\frac{\Delta Q}{2}dA.$$

In particular, with the above assumptions, solving the obstacle problem is equivalent to finding the support of the equilibrium measure associated to the $Q-$potential. This motivates the following addendum to Frostman's theorem.

Proposition 2.1.2 (Frostman, part 2)**.** Let Q be an admissible external field, K a compact subset of $\mathbb{C}$, and $d\mu = \mathbb{1}_K\frac{\Delta Q}{2}dA$. If $U^\mu + Q$ is constant q.e. on K then K is a local droplet of Q. [28, 20] (Lemma 3.2)

Proposition 2.1.2 can be applied to obtain a useful relation for solving the obstacle problem, the *coincidence equation* (Equation 2.1). To this end, we define the coincidence set

$$K_t^* = \{w \in \mathbb{C} : v_t(w) = -Q(w)\}.$$

[1] Intuitively, the equilibrium charge distribution must induce a constant potential on its support because otherwise a current would be induced, redistributing the charge.

K_t^* is compact, nonempty, and $K_t \subseteq K_t^*$. In addition, ν_t is harmonic in $\mathbb{C} \setminus K_t^*$. As a consequence, we have that

$$-Q(w) = \nu_t(w) = U^{\mu_t}(w) + c(t)$$

on K_t^*. Differentiating with respect to w, we find that

$$\frac{\partial Q}{\partial w} \doteq C_Q^{K_t}(w), \tag{2.1}$$

where $\doteq$ denotes equality on ∂K_t, and

$$C_Q^{K_t}(w) := \frac{1}{4}\int_{K_t} \frac{\Delta Q}{w - \xi} dA(\xi)$$

is $\frac{1}{2}$ the weighted Cauchy transform associated to μ_t. Note that $C_Q^{K_t}$ is analytic in K_t^c and $C_Q^{K_t}(w) = O(w^{-1})$ about ∞.

The obstacle problem is quite difficult to solve explicitly in general. However, there are a handful of cases in the literature for which this has been accomplished in the algebraic case.[9] Lemma 2.2.1 asserts a certain kind of duality between local droplets and quadrature domains: to solve the obstacle problem, it is sufficient to determine the quadrature domains associated to the local droplet. In 3.2 we described a method of recovering simply connected QDs from their quadrature function via the Faber transform. Thus this method may also be used to tackle the obstacle problem. We will also demonstrate how this duality can be applied to the uniqueness problem for quadrature domains.

2.2 Hele-Shaw Flow

For *Hele-Shaw* potentials (ΔQ is constant), we may correspond the complements of certain droplets with quadrature domains. In particular,

Lemma 2.2.1 ([20])**.** Let K be a local droplet corresponding to a Hele-Shaw potential,

$$Q(w) = |w|^2 - 2\mathrm{Re}(H(w)),$$

where $h = H'$ is a rational function. Then K^c is a disjoint union of QDs, with $h =$ the sum of their quadrature functions.

Conversely, if the complement of some compact set K is a disjoint union of QDs with $h =$ the sum of their quadrature functions, then K is a local droplet of a potential of the form

$$Q(w) = |w|^2 - 2\mathrm{Re}\left(\sum_l \mathbb{1}_{K_l} H_l(w)\right), \quad H_l(w) = c_l + \int_{w_l}^{w} h(\xi)d\xi,$$

where the $\{K_l\}_l$ are the connected components of K, w_l is some point in K_l, and c_l is a real constant.

If we consider a family of these droplets parameterized with respect to t, we arrive at the concept of a *backward Hele-Shaw chain*, or simply *Hele-Shaw chain*. Let K_{t_0} be a local droplet of charge $t_0 > 0$ of an admissible potential, Q. The backward Hele-Shaw chain associated to K_{t_0} is the family of compact sets,

$$K_t := S_t[Q_{K_{t_0}}], \quad 0 < t \leq t_0,$$

where $Q_{K_{t_0}}$ is the localization of Q to K_{t_0} and $S_t[Q_{K_{t_0}}]$ is the support of the equilibrium measure of mass t associated to $Q_{K_{t_0}}$. Several properties of the chain $\{K_t\}_t$ are enumerated below.

Lemma 2.2.2. If K_{t_0} is a local droplet of charge t_0 associated to an admissible potential Q. Then the backward Hele-Shaw chain $\{K_t\}_{0<t\leq t_0}$ associated to K_{t_0} exists and satisfies the following

1. Each K_t is a local algebraic droplet of Q.
2. The chain $\{K_t\}_t$ is monotone increasing.
3. The chain is left-continuous, $K_{t_0} = \mathrm{Cl}\left(\bigcup_{0<t<t_0} K_t\right)$.
4. If K_{t_0} has no double points, then $\exists\epsilon > 0$ such that the outer boundary of K_t has no singular points for all $t \in (t_0 - \epsilon, t_0)$.

See [15, 16, 20] for a more detailed explanation of the theory surrounding Lemma 2.2.2.

If $\{K_t\}_{0<t\leq t_0}$ is a backward Hele-Shaw chain associated to a potential Q, then we can extend the chain to $t = 0$ by defining $K_0 = \emptyset$ and K_0^* the set where the global minimum of Q is attained. That is,

Lemma 2.2.3 (remark 4.20 and corr. 4.23 in [15])**.** If $\{K_t\}_{0<t\leq t_0}$ is a backward Hele-Shaw chain associated to some potential Q, then we can extend the chain to $t = 0$ by defining $K_0 = \emptyset$ and K_0^* to be the set where the global minimum of Q is attained. Moreover, $K_0^* \subset K_t$ for each $0 < t < t_0$.

This also implies that if K is a droplet of an admissible potential Q, then K contains the set where the global minimum of Q is attained on its interior.

Example 2.2.1. As an example of the utility of this lemma, consider an unbounded domain $\Omega \in \mathrm{QD}\left(\frac{\alpha}{w}\right)$. Let K be the complement of Ω with connected components $\{K_l\}_l$. Then, by Lemma 2.2.1, K is a local droplet of the potential Q, where $Q|_{K_l}(w) = |w|^2 - 2\mathrm{Re}\left(c_l + \alpha\int_{w_l}^{w}\frac{d\xi}{\xi}\right)$, for $c_l \in \mathbb{C}$ and $w_l \in K_l$. Hence, by the previous lemma, the global minimum of Q must be attained on the interior of K. However, at such a global minimum, we must have that $\frac{\partial Q}{\partial w} = 0$. But note that

$$\left.\frac{\partial Q}{\partial w}\right|_{K_l} = \overline{w} - \frac{\alpha}{w},$$

so if w is such a global minimizer, then $\alpha = |w|^2 > 0$. Hence, we conclude that if $\mathrm{QD}\left(\frac{\alpha}{w}\right) \neq \emptyset$, then $\alpha > 0$. By the coincidence Equation, we find that

$$wC^{\Omega_*}(w) \doteq |w|^2 - \alpha \in \mathbb{R}.$$

Thus, $\mathrm{Im}(wC^{\Omega_*}(w))$ is a bounded harmonic function equal to 0 on $\partial\Omega$ so we find by the maximum modulus principle that $wC^{\Omega_*}(w) = 0$ on Ω (it has a root at 0). Hence, $|w|^2 \doteq \alpha$, and we may conclude that $\Omega = \mathbb{D}_{\sqrt{\alpha}}(0)$. In particular, $\mathrm{QD}\left(\frac{\alpha}{w}\right) = \left\{\mathbb{D}_{\sqrt{\alpha}}(0)\right\}$ if $\alpha > 0$ and is empty otherwise.

2.3 The Cauchy Projection and The Faber Transform

The Cauchy Projection

If Ω is a rectifiable domain, then there is a natural linear operator $\mathcal{C}_\Omega\cdot : C^0(\partial\Omega) \to H^\infty(\Omega)$ called the *Cauchy projection*,

$$\mathcal{C}_\Omega\left[f\right](w) := \oint_{\partial\Omega} \frac{f(\xi)}{\xi - w} d\xi, \tag{2.2}$$

The Cauchy projection has a number of useful properties, summarized in the following lemma.

Lemma 2.3.1. Fix a bounded domain Ω with a piecewise C^1 boundary.

1. If f is analytic on $\partial\Omega$,[2] then
   - $\mathcal{C}_\Omega f \in \mathcal{A}(\Omega)$,
   - $\mathcal{C}_{\Omega_*} f \in \mathcal{A}_0(\Omega_*)$,
   - $f \doteq \mathcal{C}_\Omega f + \mathcal{C}_{\Omega_*} f$.
2. If $f \in \mathcal{A}(\Omega)$, then $\mathcal{C}_\Omega f = f$ and $\mathcal{C}_{\Omega_*} f = 0$.
3. If $f \in \mathcal{A}_0(\Omega_*)$, then $\mathcal{C}_{\Omega_*} f = f$ and $\mathcal{C}_\Omega f = 0$.
4. If $f \in \mathcal{M}(\Omega)$ and extends continuously to $\partial\Omega$, then $\mathcal{C}_{\Omega_*} f \in \mathrm{Rat}_0(\Omega)$.
5. If $f \in \mathcal{M}(\Omega_*)$ and extends continuously to $\partial\Omega$, then $\mathcal{C}_\Omega f \in \mathrm{Rat}(\Omega_*)$.

The analogous results hold for unbounded Ω.

See Theorem 2.3 of [22] for a related exposition. We also refer to the Cauchy projection, $\mathcal{C}_\Omega f$, as the "analytic part" of f in Ω.

[2] meaning analytic in an open neighborhood of $\partial\Omega$

The Faber Transform

The Faber transform is a linear operator between functions analytic in the disk (or exterior disk) and functions analytic in a given simply connected domain. In this thesis, it serves as the main mechanism for explicitly relating the quadrature function to the Riemann map. See [13] §1.5.1, and [3, 10, 18], for further discussion of the Faber transform, Faber polynomials, and their properties. We will begin by defining the *interior Faber transform*.

Definition 2.3.1. Let Ω be a bounded simply connected domain with Riemann map $\varphi : \mathbb{D} \to \Omega$, normalized so that $\varphi(0) = w_0 \in \Omega$, and $\varphi'(0) > 0$. Also write $\psi := \varphi^{-1}$. The associated interior Faber transform is the operator $\Phi_\varphi : \mathcal{A}_0(\mathbb{D}_*) \to \mathcal{A}_0(\Omega_*)$ given by

$$\Phi_\varphi(f)(w) := C_{\Omega_*}\left[f \circ \psi\right](w) = \oint_{\partial\Omega_*} \frac{f \circ \psi(\xi)}{\xi - w} d\xi. \tag{2.3}$$

Its inverse $\Phi_\varphi^{-1} : \mathcal{A}_0(\Omega_*) \to \mathcal{A}_0(\mathbb{D}_*)$ is given by $\Phi_\varphi^{-1}(f) := C_{\mathbb{D}_*}\left[f \circ \varphi\right]$.

The relevant properties of Φ_φ for our purposes are the following:

- Φ_φ is a linear isomorphism $\mathcal{A}_0(\mathbb{D}_*) \to \mathcal{A}_0(\Omega_*)$.
- Φ_φ restricts to an isomorphism $\mathrm{Rat}_0(\mathbb{D}) \to \mathrm{Rat}_0(\Omega_*)$.
- If $f \in C^0(\partial\mathbb{D})$, then
$$C_{\Omega_*}\left[f \circ \psi\right] = \Phi_\varphi\left(C_{\mathbb{D}_*} f\right). \tag{2.4}$$
  Similarly, when $\partial\Omega$ is piecewise C^1 and $f \in C^0(\partial\Omega)$,
$$C_{\mathbb{D}_*}\left[f \circ \varphi\right] = \Phi_\varphi^{-1}\left(C_{\Omega_*} f\right). \tag{2.5}$$

The discussion for the *exterior Faber transform* is largely analogous to the interior case, with the primary difference being the function spaces on which it is defined, $\mathcal{A}(\mathbb{D})$. In contrast to the interior case, this allows one to consider the exterior Faber transform of polynomials, enabling the construction of Faber polynomials. We begin with the definition of the exterior transform:

Definition 2.3.2. Let Ω be an unbounded simply connected domain with Riemann map $\varphi : \mathbb{D}_* \to \Omega$, normalized so that $\varphi(\infty) = \infty$, and $\varphi'(\infty) = c > 0$. Also write $\psi := \varphi^{-1}$. The associated exterior Faber transform is the map $\Phi_\varphi : \mathcal{A}(\mathbb{D}) \to \mathcal{A}(\Omega_*)$, given by the same formula:

$$\Phi_\varphi(f)(w) := C_{\Omega_*}\left[f \circ \psi\right](w) = \oint_{\partial\Omega_*} \frac{f \circ \psi(\xi)}{\xi - w} d\xi. \tag{2.6}$$

Its inverse $\Phi_\varphi^{-1} : \mathcal{A}(\Omega_*) \to \mathcal{A}(\mathbb{D})$ is given by $\Phi_\varphi^{-1}(f) = C_{\mathbb{D}}\left[f \circ \varphi\right]$.

The corresponding properties are completely analogous:

- Φ_φ is a linear isomorphism $\mathcal{A}(\mathbb{D}) \to \mathcal{A}(\Omega_*)$.
- Φ_φ restricts to an isomorphism $\mathrm{Rat}(\mathbb{D}_*) \to \mathrm{Rat}(\Omega_*)$.
- If $f \in C^0(\partial\mathbb{D})$, then
$$C_{\Omega_*}[f \circ \psi] = \Phi_\varphi\left(C_{\mathbb{D}} f\right). \tag{2.7}$$
Similarly, when $\partial\Omega$ is piecewise C^1 and $f \in C^0(\partial\Omega)$,
$$C_{\mathbb{D}}[f \circ \varphi] = \Phi_\varphi^{-1}\left(C_{\Omega_*} f\right). \tag{2.8}$$

Faber polynomials: The Faber polynomials $\{F_n\}_{n=0}^{\infty}$ are a special polynomial basis associated to an unbounded domain Ω (and φ) are defined by

$$F_n = \Phi_\varphi(z^n) = C_{\Omega_*}[\psi^n], \quad n \geq 0.$$

Similarly, the inverse Faber polynomials $\{W_n\}_{n\geq 0}$ are defined as the inverse Faber transform of the standard monomial basis. Concretely, F_n and W_n are the polynomial parts of ψ^n and φ^n respectively. They also satisfy the following asymptotic identities

$$F_n \circ \varphi(z) = z^n + O(1), \qquad W_n \circ \psi(w) = w^n + O(1).$$

For applications throughout this thesis, an important fact is that the Faber transform of a rational function can be computed explicitly via the calculus of residues. If $w_0 \in \Omega$ and $z_0 \in \mathbb{D}$ ($\mathbb{D}_*$ for the exterior transform), then

$$\begin{aligned}
\Phi_\varphi\left(\frac{1}{(z-z_0)^n}\right)(w) &= \frac{1}{(n-1)!}\left(\frac{\varphi'(\xi)}{w-\varphi(\xi)}\right)^{(n-1)}\Bigg|_{\xi=z_0} \\
\Phi_\varphi^{-1}\left(\frac{1}{(w-w_0)^n}\right)(z) &= \frac{1}{(n-1)!}\left(\frac{\psi'(\xi)}{z-\psi(\xi)}\right)^{(n-1)}\Bigg|_{\xi=w_0}
\end{aligned} \tag{2.9}$$

These admit closed-form representations in terms of the incomplete Bell polynomials $B_{n,k}$ and the derivatives of φ:

$$\Phi_\varphi\left(\frac{1}{(z-z_0)^n}\right)(w) = \sum_{k=1}^{n} \frac{(k-1)!}{(n-1)!} \frac{B_{n,k}(\varphi'(z_0), \ldots, \varphi^{(n-k+1)}(z_0))}{(w-\varphi(z_0))^k} \tag{2.10}$$

The formula is analogous for Φ_φ^{-1}. The specific formulae for $n = 1$ and $n = 2$ are provided below.

$$\Phi_\varphi\left(\frac{1}{z-z_0}\right)(w) = \frac{\varphi'(z_0)}{w-\varphi(z_0)}, \quad \Phi_\varphi\left(\frac{1}{(z-z_0)^2}\right)(w) = \frac{\varphi''(z_0)}{w-\varphi(z_0)} + \frac{\varphi'(z_0)^2}{(w-\varphi(z_0))^2}$$
$$\Phi_\varphi^{-1}\left(\frac{1}{w-w_0}\right)(z) = \frac{\psi'(w_0)}{z-\psi(w_0)}, \quad \Phi_\varphi^{-1}\left(\frac{1}{(w-w_0)^2}\right)(z) = \frac{\psi''(w_0)}{z-\psi(w_0)} + \frac{\psi'(w_0)^2}{(z-\psi(w_0))^2} \tag{2.11}$$

$$F_1(w) = \frac{w}{c} - \frac{f_0}{c}, \qquad F_2(w) = \frac{w^2}{c^2} - \frac{2f_0}{c^2}w + \frac{f_0^2 - 2cf_1}{c^2}$$
$$W_1(z) = cz + f_0, \qquad W_2(z) = c^2z^2 + 2cf_0z + f_0^2 + 2cf_1,$$

where the f_j are the coefficients in the Laurent expansion of φ at ∞, $\varphi(z) = cz + f_0 + f_1z^{-1} + \ldots$.

The Faber transform of a rational function r can be explicitly computed via the following procedure:

1. Compute the partial fraction decomposition of r:

$$r(z) = \sum_j \frac{c_j}{(z-z_j)^{n_j}} + \sum_j b_j z^j.$$

2. Apply the linearity of the Faber transform:

$$\Phi_\varphi(r)(w) = \sum_j c_j \Phi_\varphi\left(\frac{1}{(z-z_j)^{n_j}}\right)(w) + \sum_j b_j \Phi_\varphi\left(z^j\right)(w).$$

3. Apply Equation 2.10:

$$\Phi_\varphi(r)(w) = \sum_j c_j \sum_{k=1}^{n_j} \frac{(k-1)!}{(n_j-1)!} \frac{B_{n_j,k}(\varphi'(z_j), \ldots, \varphi^{(n_j-k+1)}(z_j))}{(w-\varphi(z_j))^k} + \sum_j b_j F_j(w),$$

where each $F_j(w)$ may be computed by taking the polynomial part of ψ^j (see also [17], Chapter 18).

2.4 The Inner/Outer Factorization of Riemann Maps

We use only the simplest form of the inner–outer factorization, namely the case of bounded analytic functions that extend continuously to the boundary and have no boundary zeros. In this setting no singular inner factor appears: the inner part is a finite Blaschke product determined by the zeros, and the outer part is given by the Poisson integral formula. We record the corresponding forms on $\mathbb{D}$ and $\mathbb{D}_*$.

Lemma 2.4.1 (Inner/Outer Factorization)**.**

(a) Suppose $f \in H^\infty(\mathbb{D}) \cap C^0(\mathrm{Cl}(\mathbb{D}))$ is non-vanishing on $\partial\mathbb{D}$. Then f admits a unique factorization $f = f_{\mathrm{in}} f_{\mathrm{out}}$, where f_{in} is the finite Blaschke product consisting of the roots of f in $\mathbb{D}$, and

$$f_{\mathrm{out}}(z) = \exp\left(\frac{1}{2}\oint_{\partial\mathbb{D}} \frac{\xi + z}{\xi - z}\frac{\ln|f(\xi)|^2}{\xi} d\xi\right), \quad z \in \mathbb{D}. \tag{2.12}$$

(b) Suppose $f \in H^\infty(\mathbb{D}_*) \cap C^0(\mathrm{Cl}(\mathbb{D}_*))$ is non-vanishing on $\partial\mathbb{D}_*$. Then f admits a unique factorization $f(z) = f_{\mathrm{in}} f_{\mathrm{out}}$, where f_{in} is the finite Blaschke product consisting of the roots of f in $\mathbb{D}_*$, and

$$f_{\mathrm{out}}(z) = \exp\left(\frac{1}{2}\oint_{\partial\mathbb{D}_*} \frac{\xi + z}{\xi - z}\frac{\ln|f(\xi)|^2}{\xi} d\xi\right), \quad z \in \mathbb{D}_* \tag{2.13}$$

Part (a) is the standard inner/outer factorization in the absence of boundary zeros, so the singular inner factor is absent, and the inner part reduces to a finite Blaschke product (see [23], Theorem 1.12 for a more detailed discussion). Part (b) follows by applying part (a) to the reflection $f^\#$ on $\mathbb{D}$, then reflecting back.

For $\lambda \in \mathbb{D}$, we define the *Blaschke factor* with root λ by

$$b_\lambda(z) := \frac{\overline{\lambda}}{|\lambda|}\frac{z - \lambda}{\overline{\lambda}z - 1},$$

and set $b_0(z) := z$. We will use the following elementary properties of b_λ repeatedly: b_λ has a simple root at λ, $|b_\lambda| = 1$ on $\partial\mathbb{D}$, and $b_\lambda^\#(z) = b_\lambda(z)^{-1}$.

We now consider the consequences of this representation when f is a Riemann map. In particular, the preceding integral representation can be rewritten in terms of the Faber transform.

Theorem 2.4.1. *If φ is the Riemann map associated to a simply connected domain Ω with compact piecewise C^1 boundary such that $0 \notin \partial\Omega$, then $\varphi = \varphi_{\mathrm{in}}\varphi_{\mathrm{out}}$, where*

$$\varphi_{\mathrm{out}}(z) = C\exp\left(\Phi_\varphi^{-1}\left(C_{\Omega_*}\left[\ln|w|^2\right]\right)^\#(z)\right), \tag{2.14}$$

for some $C \in \mathbb{C}$. If Ω is bounded, then $\varphi_{\mathrm{in}}(z) = 1$ when $0 \notin \Omega$, and $\varphi_{\mathrm{in}}(z) = b_{z_0}(z)$ when $0 \in \Omega$. If Ω is unbounded, then $\varphi_{\mathrm{in}}(z) = z$ when $0 \notin \Omega$, and $\varphi_{\mathrm{in}}(z) = zb_{z_0}(z)$ when $0 \in \Omega$. z_0 is the unique root of φ.

Because φ has a simple pole at infinity in the unbounded case, it does not belong to $H^\infty(\mathbb{D}_*)$, so the classical inner-outer factorization of Lemma 2.4.1 does not apply directly. However, the normalized map $f(z) := \varphi(z)/z$ is bounded and non-vanishing on the boundary, so $f \in H^\infty(\mathbb{D}_*) \cap C^0(\mathrm{Cl}(\mathbb{D}_*))$. Therefore, we define the factorization of φ by decomposing $f = f_{\mathrm{in}} f_{\mathrm{out}}$ via Lemma 2.4.1. We define the *outer factor* of the Riemann map as $\varphi_{\text{out}} := f_{\text{out}}$ and the *extended inner factor* as $\varphi_{\text{in}}(z) := z f_{\text{in}}(z)$. The proof of Theorem 2.4.1 is in Appendix A.2.

The Power Factorization

For certain classes of weighted quadrature domains, a factorization of the form $\varphi = \varphi_{\text{in}}\varphi_{\text{out}}$, where $\varphi_{\text{out}} = f^{\frac{1}{a}}$ for some $a > 0$ and analytic function f, is more suitable. We state the analog of Theorem 2.4.1 for an outer factor of this form here.

Theorem 2.4.2. *If φ is the Riemann map associated to a simply connected domain Ω with compact piecewise C^1 boundary, then $\varphi = \varphi_{\text{in}}\varphi_{\text{out}}$, where*

$$\varphi_{\text{out}}^a = \Phi_\varphi^{-1}\left(C_{\Omega_*}\left[|w|^{2a}\left(\frac{\varphi_{\text{in}} \circ \psi(w)}{w}\right)^a\right]\right)^{\#} + C, \tag{2.15}$$

for some $C \in \mathbb{C}$. If Ω is bounded, then $\varphi_{\text{in}}(z) = 1$ when $0 \notin \Omega$, and $\varphi_{\text{in}}(z) = b_{z_0}(z)$ when $0 \in \Omega$. If Ω is unbounded, then $\varphi_{\text{in}}(z) = z$ when $0 \notin \Omega$, and $\varphi_{\text{in}}(z) = z b_{z_0}(z)$ when $0 \in \Omega$. Here, z_0 is the unique root of φ.

The proof is in Appendix A.2. It is worth noting that, redefining C appropriately, Equation 2.15 is equivalent to

$$\frac{\varphi_{\text{out}}^a(z) - 1}{a} = \Phi_\varphi^{-1}\left(C_{\Omega_*}\left[\frac{|w|^{2a} - 1}{a}\left(\frac{\varphi_{\text{in}} \circ \psi(w)}{w}\right)^a\right]\right)^{\#}(z) + C.$$

So sending, $a \to 0^+$, we obtain

$$\ln \circ \varphi_{\text{out}}(z) = \Phi_\varphi^{-1}\left(C_{\Omega_*}\left[\ln |w|^2\right]\right)^{\#}(z) + C,$$

which is equivalent to Equation 2.14.

Chapter 3

QUADRATURE DOMAINS

Given a rectifiable domain Ω and an integrable test class $\mathcal{F} = \mathcal{F}(\Omega)$ of functions analytic in Ω and continuous on $\mathrm{Cl}(\Omega)$, we informally say that Ω is a quadrature domain *with respect to* $\mathcal{F}$ if it satisfies a Green's theorem-like identity of the form

$$\int_{\Omega} f\,dA = \oint_{\partial\Omega} f(w)h(w)\,dw, \quad f \in \mathcal{F}(\Omega), \tag{3.1}$$

where h is a rational function with poles only in Ω, referred to as a *quadrature function*. The poles of h are referred to as *quadrature nodes*. The most well-known example of such a domain is the open disk $\Omega = \mathbb{D}_r(w_0)$. In particular, if $f \in \mathcal{A}(\mathbb{D}_r(w_0))$ then, by Green's theorem and the fact that $|w - w_0|^2 \doteq r^2$, we find

$$\begin{aligned} \int_{\mathbb{D}_r(w_0)} f\,dA(w) &= \oint_{\partial\mathbb{D}_r(w_0)} f(w)\overline{w}\,dw \\ &= \oint_{\partial\mathbb{D}_r(w_0)} f(w)\left(\frac{r^2}{w - w_0} + \overline{w_0}\right)dw \\ &= \oint_{\partial\mathbb{D}_r(w_0)} f(w)\frac{r^2}{w - w_0}\,dw \end{aligned}$$

Hence, $\Omega = \mathbb{D}_r(w_0)$ satisfies an identity in the form of Equation 3.1, with $h(w) = \frac{r^2}{w-w_0}$. Aharonov & Shapiro [1] demonstrated the converse statement: disks are the only finitely connected bounded domains which satisfy the mean value property.

Note that if Ω is bounded, then Equation 3.1 is equivalent to the existence of a finite collection of $a_k \in \Omega$, $c_k \in \mathbb{C}$, and $m_k \in \mathbb{N}$ such that

$$\int_{\Omega} f\,dA = \sum_k c_k f^{(m_k)}(a_k), \quad f \in \mathcal{F}(\Omega). \tag{3.2}$$

This is commonly referred to as a *quadrature identity*. These two formulations are related via the residue theorem, whence we obtain the formula $h(w) = \sum_k \frac{c_k m_k!}{(w-a_k)^{m_k+1}}$. Returning to the prior example of $\Omega = \mathbb{D}_r(w_0)$, we recover the classical mean value property for the disk,

$$\int_{\mathbb{D}_r(w_0)} f\,dA = r^2 f(w_0).$$

We will now formally define quadrature domains along the lines of [13], beginning with the case in which Ω is bounded.

Definition 3.0.1 (Bounded quadrature domain)**.** A bounded rectifiable domain $\Omega \subset \mathbb{C}$ is a quadrature domain if there exists $h \in \mathrm{Rat}_0(\Omega)$ such that Equation 3.1 holds for all $f \in \mathcal{A}(\Omega)$.

However, if Ω is an unbounded domain, then $\mathcal{A}(\Omega)$ is not an integrable test class (even in principal value). Hence, a slightly smaller test class is necessary for a consistent definition of unbounded quadrature domains.

Definition 3.0.2 (Unbounded quadrature domain)**.** An unbounded rectifiable domain $\Omega \subset \widehat{\mathbb{C}}$ is a quadrature domain if there exists $h \in \mathrm{Rat}(\Omega)$ such that Equation 3.1 holds for all $f \in \mathcal{A}_0(\Omega)$.

By $\Omega \in \mathrm{QD}(h)$, we denote that Ω is a bounded or unbounded QD with quadrature function h. We furthermore write $\Omega \in \mathrm{QD}$ if there exists h such that $\Omega \in \mathrm{QD}(h)$. *We will always require that* $\Omega = \mathit{Int}(\mathit{Cl}(\Omega))$ because otherwise arbitrarily many QDs with the same quadrature function could be constructed via deletion of subsets of measure zero.

3.1 Basic Properties

The following discussion (cf. [13, 20]) summarizes several important properties/characterizations of quadrature domains.

Proposition 3.1.1. Let Ω be a rectifiable domain. Then the following are equivalent

1. $\Omega \in \mathrm{QD}$,
2. $C^{\Omega}|_{\Omega_*}$ extends to a rational function on $\widehat{\mathbb{C}}$,
3. Ω admits a Schwarz function.

A *Schwarz function* $S : \mathrm{Cl}(\Omega) \to \widehat{\mathbb{C}}$ is a meromorphic function extending continuously to the boundary such that $S(w) \doteq \overline{w}$.[1] C^{Ω} denotes the *Cauchy transform*

$$C^{\Omega}(w) = \int_{\Omega} \frac{dA(\xi)}{w - \xi}.$$

Furthermore, if $\Omega \in \mathrm{QD}(h)$, then its Schwarz function satisfies the identity [8]

$$S(w) = h(w) + C^{\Omega_*}(w). \tag{3.3}$$

The boundary regularity of quadrature domains depends crucially on the existence of a Schwarz function; in particular, a *local Schwarz function*. This follows from the celebrated Sakai regularity theorem (3.1.1). We say that S is a local Schwarz function for Ω at $w_0 \in \partial\Omega$ if there exists $\epsilon > 0$ such that $S \in \mathcal{A}(\mathbb{D}_\epsilon(w_0) \cap \mathrm{Cl}(\Omega))$ and $S(w) = \overline{w}$ on $\mathbb{D}_\epsilon(w_0) \cap \partial\Omega$. The Sakai regularity theorem states:

[1] Recall that $\doteq$ denotes equality on $\partial\Omega$.

Theorem 3.1.1 (Sakai Regularity [30])**.** *If S is a* local Schwarz function *at a singular point w_0 of $\partial\Omega$, then w_0 is either a (conformal)* cusp*, a* double point*, or a* degenerate point*.*

See [20], §3.2 for a more formal discussion of Sakai regularity and the relevant nomenclature (cusp, double point, degenerate point, local Schwarz function, etc.). An immediate consequence of Sakai's theorem is the following regularity theorem for QDs:

Corollary 3.1.1. If $\Omega \in \mathrm{QD}$, then $\partial\Omega$ has finitely many singular points, and each is a cusp or a double point.

An important consequence of Corollary 3.1.1 is that the boundary of a quadrature domain is piecewise C^1. This level of boundary regularity implies a number of nice properties for function spaces on Ω (e.g. Lemma 2.3.1).

3.2 The Inverse and Direct Problems for Simply Connected QDs

A central problem in the theory of quadrature domains is that of reconstructing a QD given its quadrature function. This is referred to as the *inverse problem* for quadrature domains. Complementary to this is the *direct problem*, which is concerned with determining the quadrature function associated to a given quadrature domain.

For simply connected domains, these problems may be reduced to relating the quadrature function of the domain and its Riemann map. A classic result in the theory of quadrature domains provides a characterization of simply connected QDs in terms of the associated Riemann map, which we state here.

Theorem 3.2.1 ([1])**.** *A simply connected domain is a quadrature domain if and only if its Riemann map extends to a rational function.*

Hence, for simply connected QDs, the inverse and direct problems reduce to the problem of determining the relationship between the finitely many poles and coefficients of the quadrature function, and the finitely many poles and coefficients of the Riemann map. Formulae relating the Riemann map of a given simply connected QD with its quadrature identity have appeared in various forms going as far back as Davis [8] (1974) in the case of bounded classical QDs. More recently, Ameur, Helmer, and Tellander (2021) apply a form of this identity ([2] Equation 2), referred to as the "master formula", to the uniqueness problems for bounded classical QDs. This "master formula" represents the Riemann map in terms of the quadrature measure $\mu = \bar{\partial} h$:

$$\varphi(z) = \overline{\varphi_*\mu}\left(\frac{z}{\overline{\varphi'(\lambda)}(1 - z\bar{\lambda})}\right), \tag{3.4}$$

where $\varphi_*\mu$ is the pushforward measure.

Chang [6] considers an essentially equivalent formula in terms of the *Faber transform*, an isomorphism of analytic functions on the disk with those on another bounded simply connected domain. We provide generalizations of this approach to unbounded domains (Theorem 3.2.3) and some classes of weighted domains (§4, §5, §6). After fixing some conventions and notation regarding the Riemann map, we will proceed with a brief proof of Theorem 3.2.1, followed by an exposition of the method for classical BQDs and UQDs.

Let Ω be a bounded simply connected domain. The Riemann mapping theorem asserts that Ω is conformally equivalent to the disk, and that the associated conformal map is unique up to a rotation and recentering. In particular, for each $w_0 \in \Omega$ there exists a unique biholomorphism $\varphi : \mathbb{D} \to \Omega$ for which $\varphi(0) = w_0$ and $\varphi'(0) > 0$, so that

$$\varphi(z) = w_0 + \varphi'(0)z + O(z^2)$$

We refer to φ as *the* Riemann map associated to Ω and w_0. On the other hand, when Ω is unbounded, there exists a unique biholomorphism $\varphi : \mathbb{D}_* \to \Omega$ for which $\varphi(\infty) = \infty$ and $\varphi'(\infty) > 0$, so that

$$\varphi(z) = cz + f_0 + O(z^{-1}).$$

In this case, too, we refer to φ as *the* Riemann map associated to Ω.

Solving the Inverse and Direct Problems via the Faber Transform

A central goal of this thesis is to demonstrate a method of reconstructing the Riemann map associated to a simply connected QD from its quadrature function and vice-versa (i.e. to solve the inverse and direct problems for simply connected QDs). Theorems 3.2.2 and 3.2.3 provide characterizations of this method for bounded and unbounded domains respectively. We denote by Φ_φ the *Faber transform* associated to the Riemann map φ.

Theorem 3.2.2 ([13], Theorem 1.5)**.** *Let $\Omega \in QD(h)$ be a bounded and simply connected domain with quadrature function $h \in \mathrm{Rat}_0(\Omega)$ and Riemann map $\varphi : \mathbb{D} \to \Omega$. In this case,*

$$h = \Phi_\varphi\left(C_{\mathbb{D}_*}\left[\varphi^\#\right]\right) \tag{3.5}$$

and

$$\varphi = \varphi(0) + \Phi_\varphi^{-1}(h)^\#. \tag{3.6}$$

In particular, Equations 3.5 and 3.6 respectively solve the direct and inverse problems for BQDs. Note that $C_{\mathbb{D}_*}\varphi^\# = \varphi^\# - \overline{\varphi(0)}$, so $h = \Phi_\varphi\left(\varphi^\# - \overline{\varphi(0)}\right)$ is an equivalent form of Equation 3.5. Theorem 3.2.3 covers the analogous result for UQDs.

Theorem 3.2.3 ([13], Theorem 1.6)**.** *Let $\Omega \in QD(h)$ be an unbounded and simply connected domain with quadrature function $h \in \mathrm{Rat}(\Omega)$ and Riemann map $\varphi : \mathbb{D}_* \to \Omega$. In this case,*

$$h = \Phi_\varphi \left(C_{\mathbb{D}} \left[\varphi^\# \right] \right) \tag{3.7}$$

and

$$\varphi(z) = cz + \Phi_\varphi^{-1}(h)^\#(z). \tag{3.8}$$

Note that $C_{\mathbb{D}} \left[\varphi^\#(z) \right] = \varphi^\#(z) - cz^{-1}$, so $h = \Phi_\varphi \left(\varphi^\#(z) - cz^{-1} \right)$ is an equivalent form of Equation 3.7.

For a more detailed discussion and proofs of Theorems 3.2.2 and 3.2.3, see Graven & Makarov [13]. With these results established, we are now prepared to state and prove the main theorem of this chapter: Theorem 3.3.1, which gives a complete classification of one-point quadrature domains. This classification includes *unbounded* one-point QDs - a family with far richer structure than the classical bounded case.

3.3 Classification of One-Point QDs

We are now prepared to consider the problem of classifying simply connected one-point quadrature domains. That is, simply connected $\Omega \in \mathrm{QD}\left(\frac{\alpha}{w - w_0}\right)$ for some $\alpha \in \mathbb{C}$ and $w_0 \in \mathbb{C}$. That is, we're interested in classifying domains Ω admitting a quadrature identity of the form

$$\int_\Omega f \, dA = \alpha f(w_0)$$

While the case in which $\alpha \geq 0$ has been extensively studied (see, e.g. [9]), the $\alpha \in \mathbb{C}$ case has received little attention. It is well-known that if $\alpha = 0$ then Ω is an exterior disk centered at the origin (e.g. [20]). It is worth noting that, while both BQDs and UQDs generically exist when $\alpha > 0$, this is not true when $\alpha \ngeq 0$, in which case there are no BQDs (this follows straightforwardly from applying the quadrature identity to $1 \in \mathcal{A}(\Omega)$). The central result of this section is the following classification theorem:

Theorem 3.3.1. *Take $\alpha \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$ and $w_0 \neq 0$. Then*

$$QD\left(\frac{\alpha}{w - w_0}\right) \neq \emptyset \qquad \Longleftrightarrow \qquad |w_0|^2 + 2Re\,(\alpha) > 2|\alpha|.$$

In this case, there exists $t_* \in (0, \infty)$ such that

1. *$QD\left(\frac{\alpha}{w - w_0}\right) = \{\Omega_t\}_{0 < t \leq t_*}$, a continuous monotone (2.2.2) family of simply connected domains (where $t = A(\Omega_t)$);*

2. *$\partial\Omega_t$ is smooth for all $0 < t < t_*$ and the family terminates with the formation of a $(3,2)$ cusp at $t = t_*$.*

In the course of the proof of 3.3.1, we also obtain a characterization of the Riemann map associated to each Ω_t:

Corollary 3.3.1. If $\Omega \in \mathrm{QD}\left(\frac{\alpha}{w-w_0}\right)$ is unbounded then it is simply connected and its Riemann map is given by Equation 3.10.

We begin by simplifying the situation via a change of variables: Note that if $a, b \in \mathbb{C}$ and $a \neq 0$, then

$$\begin{aligned} \Omega \in \mathrm{QD}(h) \qquad &\Longleftrightarrow \qquad a\Omega + b \in \mathrm{QD}\left(\overline{a}h\left(\frac{w-b}{a}\right)\right), \\ \Omega \in \mathrm{QD}(h) \qquad &\Longleftrightarrow \qquad a\Omega + b \in \mathrm{QD}\left(\overline{a}h\left(\frac{w-b}{a}\right) + \overline{b}\right), \end{aligned} \tag{3.9}$$

when Ω is bounded and unbounded respectively. To see this note that if we set $g(w) = aw + b$, then

$$\int_{g(\Omega)} f\,dA = \int_{\Omega} f \circ g|g'|^2 dA = |a|^2 \oint_{\partial\Omega} f \circ g(w)h(w)dw = \overline{a} \oint_{\partial g(\Omega)} f(w)\, h \circ g^{-1}(w)dw.$$

The unbounded case is analogous, except one must take into account the Cauchy principal value.

Applying this result (Equation 3.9) with $a = \frac{|w_0|}{w_0}$ and $b = 0$, yields $\Omega \in \mathrm{QD}\left(\frac{\alpha}{w-w_0}\right)$ if and only if $\frac{|w_0|}{w_0}\Omega \in \mathrm{QD}\left(\frac{\alpha}{w-|w_0|}\right)$. So we can WLOG assume that $w_0 > 0$ (unless $w_0 = 0$, in which case α must be positive and $\Omega = \mathbb{D}_{\sqrt{\alpha}}(0)$ [Example 2.2.1]). We begin with a discussion of the case in which α is real.

One-Point QDs for Real α

While the family of one-point QDs with positive quadrature constant α is well understood, much less is known concerning the case in which $\alpha < 0$. We review the positive case first.

The positive case ($\alpha > 0$):

Here it is instructive to consider the problem from the perspective of Hele-Shaw flow. In particular, by Lemma 2.2.1 if $\Omega \in \mathrm{QD}\left(\frac{\alpha}{w-w_0}\right)$ is unbounded and simply connected, then its complement K is a local droplet of the potential

$$Q(w) = |w|^2 - 2\mathrm{Re}(\alpha \ln(w - w_0)).$$

Thus one might expect a $1-$parameter family of QDs $\{\Omega_t\}_t$, parameterized by the measure of the droplets in the corresponding Hele-Shaw chain. This turns out to be the case: if $\alpha > 0$ then there is a unique $1-$parameter family of simply connected domains

$$\{\Omega_t\}_{0<t\leq t_*} = \mathrm{QD}\left(\frac{\alpha}{w - w_0}\right) \bigcap \left\{\Omega \subset \widehat{\mathbb{C}} : \Omega \text{ is unbounded}\right\},$$

where $t_* = w_0(w_0 + 2\sqrt{\alpha})$. This family of domains may be represented by a one-parameter family of Riemann maps, $\{\varphi_t : \mathbb{D}_* \to \Omega_t\}_t$, given explicitly by

$$\varphi_t(z) = cz\frac{z - z_0 + \frac{w_0}{c}\frac{|z_0|^2-1}{|z_0|^2}}{z - \overline{z_0}^{-1}}, \tag{3.10}$$

where $c = c(t)$ is the conformal radius of Ω_t, and $z_0 = z_0(c)$ is the unique real number ≥ 1 for which

$$0 = c^2 z_0^4 - cw_0 z_0^3 - \alpha z_0^2 - cw_0 z_0 + w_0^2, \text{ and } \varphi_t(1) < w_0. \tag{3.11}$$

Such a z_0 can be shown to exist if and only if $0 < c \leq w_0 + \sqrt{\alpha}$ (corresponding to $0 < t \leq w_0(w_0 + 2\sqrt{\alpha})$). c and t satisfy

$$0 = t^4 + (2\alpha - c^2)t^3 + (\alpha^2 + 2c^2(3w_0^2 - \alpha))t^2 + c^2 w_0^2(4\alpha + w_0^2)t + 8c^6 w_0^2 \tag{3.12}$$

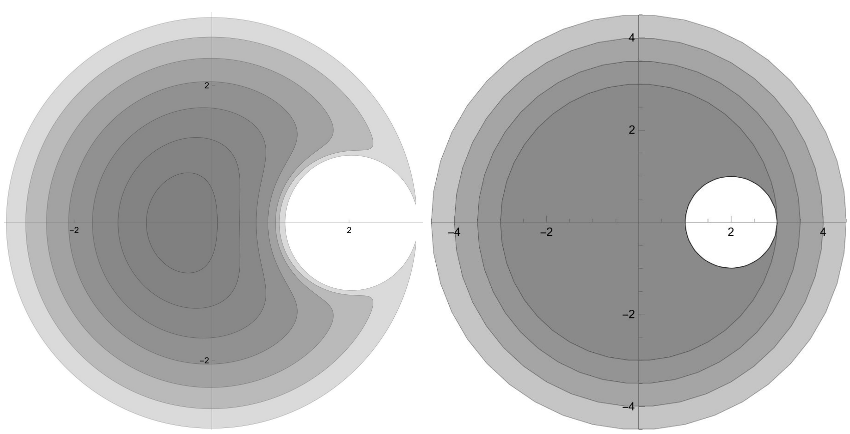

Figure 3.1: $\Omega_t \in \mathrm{QD}\left(\frac{1}{w-2}\right)$ (complements of shaded regions) for increasing t up to the critical time (left) and after the critical time (right).

On the other hand when $t > t_*$ the unique *bounded* family $\{\Omega_t\}_{t>t_*}$ is constant. It is well-known that $\Omega_t = \mathbb{D}_{\sqrt{\alpha}}(w_0)$. From the perspective of Hele-Shaw flow, this can be understood as a two-phase flow for which $K_t = \Omega_t^c$ is initially simply connected, before forming a double-point at some critical time t_*, after which K_t becomes doubly connected of the form $\mathbb{D}_{r(t)}(0) \setminus \mathbb{D}_{\sqrt{\alpha}}(w_0)$ (see Figure 3.1). Note that for $t \geq t_*$, K_t^c actually contains two components, one being our one-point BQD, and the other a null QD.

The negative case ($\alpha < 0$):

The situation when $\alpha < 0$ is in many ways the same as the $\alpha > 0$ case. In particular, there exists $0 < t_* < \infty$ and a simply connected 1−parameter family $\{\Omega_t\}_{0<t\leq t_*} = \mathrm{QD}\left(\frac{\alpha}{w-w_0}\right)$ if and only if $-w_0^2 < 4\alpha < 0$. They too are characterized by Equations 3.10 and 3.11. The two cases are distinguished, however, by their behaviour at the critical time. In particular, while Ω_{t_*} forms a double point when $\alpha > 0$, a $(3,2)$−cusp develops as $t \to t_*$ when $\alpha < 0$, after which no such Ω_t exists. Moreover c_* is the middle root of the cubic $8c^3 - 15w_0c^2 + 6(w_0^2 + 4\alpha)c + \frac{(w_0^2-\alpha)(w_0^2+4\alpha)}{w_0}$, which is given explicitly by

$$c_* = \frac{w_0}{8}\left(5 - 6\sqrt{1-\beta}\sin\left(\frac{1}{3}\sin^{-1}\left(\frac{\frac{3}{32}\beta^2 - 3\beta - 1}{(1-\beta)^{\frac{3}{2}}}\right)\right)\right), \quad \beta = \alpha\left(\frac{8}{3w_0}\right)^2.$$

t_* can then be determined from Equation 3.12 (see Figure 3.2).

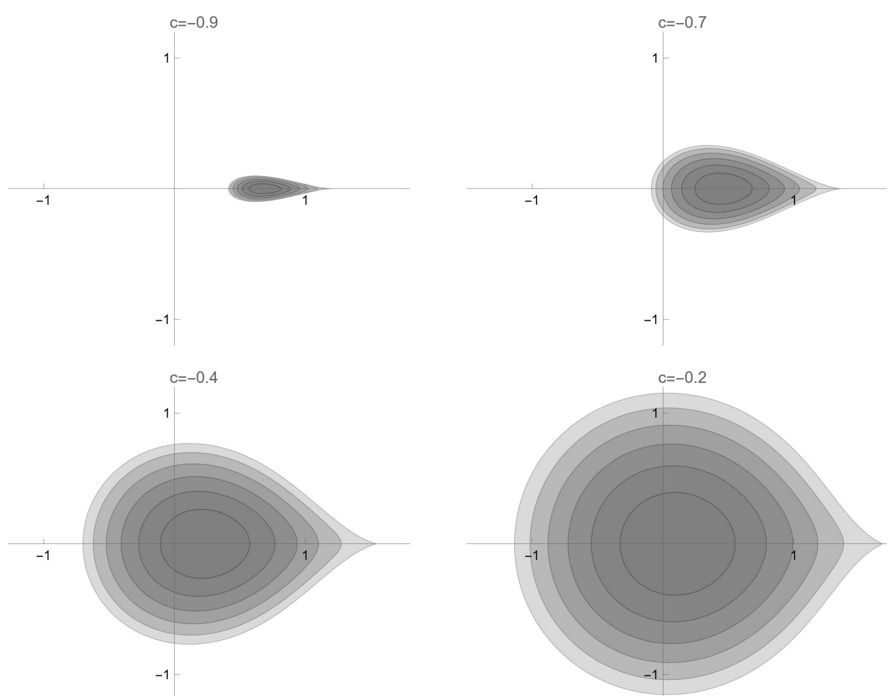

Figure 3.2: $\Omega_t \in \mathrm{QD}\left(\frac{\alpha}{w-2}\right)$ (complements of shaded regions) for $\alpha \in \{-.9, -.7, -.4, -.2\}$ for increasing t up to the critical time.

We now shift our consideration to the more general situation of $\alpha \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$, in which case Ω is necessarily unbounded when it exists. Here we will see that such domains exist precisely when α lies within the parabolic region characterized by $|w_0|^2 + 2\mathrm{Re}(\alpha) > 2|\alpha|$. In addition, we will see that Ω_t (as defined above) ceases to exist for t sufficiently large. We're now prepared to prove the main theorem of the section.

Proof of Theorem 3.3.1

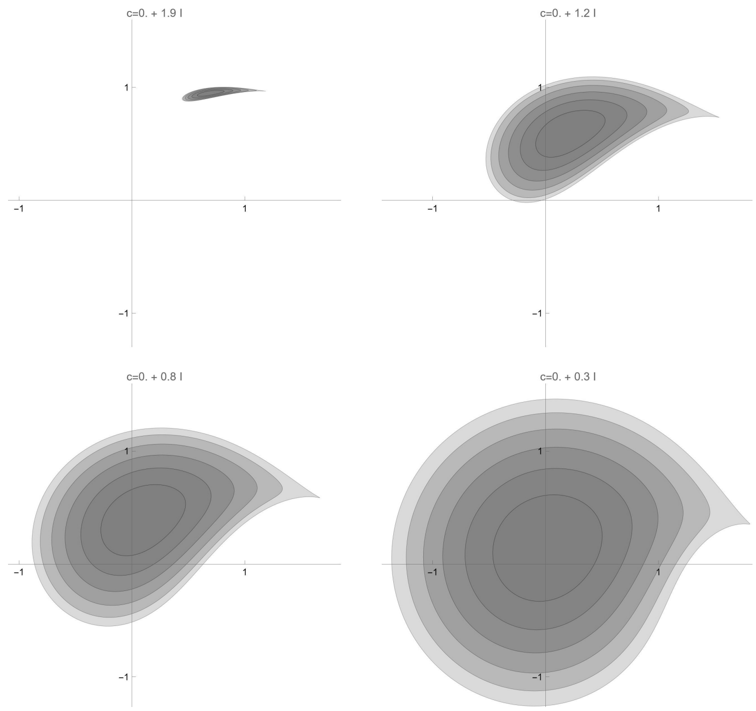

Figure 3.3: Ω_t^c for $\alpha \in \{1.9i, 1.2i, .8i, .3i\}$ and $w_0 = 2$ (shaded) for increasing t up to the critical time.

First and foremost, we establish that $\Omega \in \mathrm{QD}\left(\frac{\alpha}{w-w_0}\right)$ is simply connected: This is an immediate consequence of the fact that Ω is a quadrature domain of order $d_\Omega = \deg\left(\frac{\alpha}{w-w_0}\right) = 1$, so Lee & Makarov's topological bound ([20] §1.4) implies Ω is simply connected. Next, by Equation 3.9, $\Omega \in \mathrm{QD}\left(\frac{\alpha}{w-w_0}\right)$ if and only if $\frac{2}{w_0}\Omega \in \mathrm{QD}\left(\frac{4\alpha|w_0|^{-2}}{w-2}\right)$. That is, it will be sufficient to consider $\Omega \in \mathrm{QD}\left(\frac{\alpha}{w-2}\right)$.

Existence Criterion

The general approach here is to show

1. $\mathrm{QD}\left(\frac{\alpha}{w-2}\right)$ is non-empty $\iff$ a certain potential Q has a local minimum,

2. Q has a local minimum $\iff$ $2 + \mathrm{Re}(\alpha) > |\alpha|$.

(1) Suppose there exists an unbounded $\Omega \in \mathrm{QD}\left(\frac{\alpha}{w-2}\right)$. Then, by Lemma 2.2.1, $K := \Omega^c$ is a local droplet of the potential

$$Q(w) = |w|^2 - 2\mathrm{Re}(\alpha \ln(w - w_0)) + \sum_{j=1}^{n} c_j \mathbb{1}_{K^{(j)}}, \quad K = \bigsqcup_{j=1}^{n} K^{(j)},$$

where the $K^{(j)}$ are the connected components of K. Let $\{K_t\}_{t=0}^{\mathrm{A}(K)}$ be the backwards Hele-Shaw chain associated to K, so that $K_0^* \neq \emptyset$ (2.2.2). However, K_0^* is the locus of minima of Q so it must have a local minimum.

On the other hand, suppose $Q(w) = |w|^2 - 2\mathrm{Re}(\alpha \ln(w - 2))$ has a local minimum, so it admits a local droplet K_t for some $t > 0$. Then $\Omega := K_t^c \in \mathrm{QD}\left(\frac{\alpha}{w-2}\right)$ (by 2.2.1).

(2) We now show that $Q(w) = |w|^2 - 2\mathrm{Re}(\alpha \ln(w - 2))$ admits a local minimum if and only if $2 + \mathrm{Re}(\alpha) > |\alpha|$, and that this local minimum is unique when it exists. Q may have a local minimum only if $\overline{w} = \frac{\alpha}{w-2}$. Taking this equation with its conjugate, we arrive at the following set of all candidate minima:

$$w = \frac{1}{2}\left(2 + i\mathrm{Im}(\alpha) \pm \sqrt{4 + 4\mathrm{Re}(\alpha) - \mathrm{Im}^2(\alpha)}\right).$$

If $4 + 4\mathrm{Re}(\alpha) - \mathrm{Im}^2(\alpha) < 0$, then

$$\overline{w} - \frac{\alpha}{w-2} = \pm i\sqrt{\mathrm{Im}^2(\alpha) - 4\mathrm{Re}(\alpha) - 4} \neq 0,$$

so no solutions exist in this case. If $4 + 4\mathrm{Re}(\alpha) - \mathrm{Im}^2(\alpha) = 0$, then our unique candidate solution is $w = 1 + i\dfrac{\mathrm{Im}(\alpha)}{2}$. However, expanding in series about this point demonstrates that it is not a minimum.

Finally, if $4+4\mathrm{Re}(\alpha)-\mathrm{Im}^2(\alpha) > 0$, then its easy to show that $\frac{\partial Q}{\partial w} = 0$, so we have candidate minima only in this case. Local analysis demonstrates that $w = \frac{1}{2}\left(2 + i\mathrm{Im}(\alpha) + \sqrt{4 + 4\mathrm{Re}(\alpha) - \mathrm{Im}^2(\alpha)}\right)$ is not, in fact, a minimum. So, finally, we'll check that $w = \frac{1}{2}\left(2 + i\mathrm{Im}(\alpha) - \sqrt{4 + 4\mathrm{Re}(\alpha) - \mathrm{Im}^2(\alpha)}\right)$ is a local minimum. For a root of $\frac{\partial Q}{\partial w}$ to be a local minimum, it is sufficient to check that $\Delta Q > 4\left|\frac{\partial^2 Q}{\partial w^2}\right|$ or, equivalently, $|\alpha| < |w-2|^2$. By assumption,

$$-1 < \frac{\mathrm{Re}(\alpha) + 1}{\sqrt{4 + 4\mathrm{Re}(\alpha) - \mathrm{Im}^2(\alpha)}},$$

so

$$|\alpha|^2 = \mathrm{Re}^2(\alpha) + \mathrm{Im}^2(\alpha) < \left(\mathrm{Re}(\alpha) + 2 + \sqrt{4 + 4\mathrm{Re}(\alpha) - \mathrm{Im}^2(\alpha)}\right)^2,$$

thus, noting that $\mathrm{Re}(\alpha) > -1$,

$$|\alpha| < \mathrm{Re}(\alpha) + 2 + \sqrt{4 + 4\mathrm{Re}(\alpha) - \mathrm{Im}^2(\alpha)} = \frac{\mathrm{Im}^2(\alpha) + \left(2 + \sqrt{4 + 4\mathrm{Re}(\alpha) - \mathrm{Im}^2(\alpha)}\right)^2}{4} = |w-2|^2$$

The fact that $4 + 4\mathrm{Re}(\alpha) < \mathrm{Im}^2(\alpha) \iff 2 + \mathrm{Re}(\alpha) > |\alpha|$ completes the proof of this second claim. We conclude that $\mathrm{QD}\left(\frac{\alpha}{w-2}\right)$ is non-empty $\iff 2 + \mathrm{Re}(\alpha) > |\alpha|$.

The Riemann Map

Fix $\alpha \not\geq 0$ so Ω is unbounded (and simply connected, by above), and let φ be the Riemann map associated to Ω. Then by Equation 3.8,

$$\varphi(z) = cz + \Phi_\varphi^{-1}\left(\frac{\alpha}{w-2}\right)^{\#}(z) = cz + \left(\frac{\alpha\psi'(2)}{z - \psi(2)}\right)^{\#} = cz + \frac{\overline{\alpha}}{\overline{\varphi'(z_0)}}\frac{1}{z^{-1} - \overline{z_0}} = cz\frac{z - z_1}{z - \overline{z_0}^{-1}}$$

for some $z_1 \in \mathbb{C}$. Now recall that $2 = \varphi(z_0)$, which implies $z_1 = z_0 - \frac{2}{c}\frac{|z_0|^2-1}{|z_0|^2}$, so

$$\varphi(z) = cz\frac{z - z_0 + \frac{2}{c}\frac{|z_0|^2-1}{|z_0|^2}}{z - \overline{z_0}^{-1}}.$$

In particular we have shown that if $\Omega \in \mathrm{QD}\left(\frac{\alpha}{w-2}\right)$ for some $\alpha \not\geq 0$, then Ω may be represented by a Riemann map of the above form.

Existence and Uniqueness of the One-Parameter Family

The fact that $\mathrm{QD}\left(\frac{\alpha}{w-2}\right)$ is nonempty when $2 + \mathrm{Re}(\alpha) > \alpha$ was established in 3.3. Thus the existence of the continuous monotone family $\{\Omega_t\}_t$ follows directly from the existence of a backward Hele-Shaw chain (2.2.2). For uniqueness, suppose $\exists \Omega_t$ and Ω_t' with $A(\Omega_t) = A(\Omega_t') = t > 0$. Then by lemmas 2.2.1 and 2.2.3, the complements of both of these domains, K_t and K_t', are local droplets of Q, which has a unique local minimum in $K_t \cap K_t'$. If we localize to an open nbhd of $K_t \cup K_t'$ then, by Frostman's theorem 2.1.1, $K_t = K_t'$ q.e, so $\Omega_t = \Omega_t'$.

Now all we need establish is the existence of such a maximal t_*. Firstly, Equation 3.7 gives

$$\frac{\alpha}{w-2} = h(w) = \Phi_\varphi\left(C_{\mathbb{D}}\varphi^{\#}\right)(w) = \frac{(cz_0|z_0|^2 - 2)(c\overline{z_0} - 2)}{|z_0|^2(w-2)},$$

(we compute using partial fractions and Equation 2.9). This implies

$$\alpha|z_0|^2 = (cz_0|z_0|^2 - 2)(c\overline{z_0} - 2). \tag{3.13}$$

Then, setting $z_0 = c^{-1}re^{i\theta}$, we find that $0 = r^4 - 2e^{i\theta}r^3 - \alpha r^2 - 2c^2re^{-i\theta} + 4c^2$. Then, considering the asymptotics as $c \to \infty$, we see that either $r \in O(c^{\frac{2}{3}})$ or $r \in O(1)$. Thus $|z_0| \lesssim c^{-\frac{1}{3}}$, so $z_0 \xrightarrow{c\to\infty} 0$ but $|z_0| > 1$, so no solution exists for c sufficiently large. Thus there exists a maximal $t_* > 0$.

Cusp Formation

We would like to show that (a) $\partial\Omega_{t_*}$ contains a single $(3,2)$ cusp and (b) $\partial\Omega_t$ is smooth whenever $0 < t < t_*$. Recall that Ω has a Riemann map in the form of Equation 3.10. Thus for (a) all we need show is that if $\partial\Omega_t = f_t(\partial\mathbb{D})$ (where $f_t(z) = z\frac{z-z_0-\alpha}{z-\overline{z_0}^{-1}}$ for some $\alpha = \alpha(t) \in \mathbb{R}$ and $z_0 = z_0(t) \in \mathbb{D}_*$) is a continuous family of bounded curves and there exists some maximal $t_* > 0$ such that f_t is univalent in $\mathbb{D}_*$, then $\partial\Omega_{t_*}$ contains a single $(3,2)$ cusp. In particular, for (a) it is sufficient to show that

1. If $f_t'(z_*) = 0$ for some $z_* \in \partial\mathbb{D}$, then $f_t(\partial\mathbb{D})$ has a $(3,2)-$cusp at $f_t(z_*)$.

2. If f_t continuously transitions out of univalence at $t = t_*$, then there exists $z_* \in \partial\mathbb{D}$ such that $f_t'(z_*) = 0$.

(1) $f_t'(z) = z_0\frac{z(z\overline{z_0}-2)+z_0+\alpha}{(z\overline{z_0}-1)^2}$, so $0 = f_t'(z_*) = z_*(z_*\overline{z_0}-2) + z_0 + \alpha$. Considering this equation along with its conjugate, we obtain $z_0 + \alpha = 2 - \overline{z_0}$, so $f_t(z) = z\frac{z-2+\overline{z_0}}{z-\overline{z_0}^{-1}}$. Once again setting $f_t'(z_*) = 0$, and solving for z_*, we find that $z_* = 1$ or $z_* = \frac{2-\overline{z_0}}{\overline{z_0}}$. However, in the latter case, we find that $\mathrm{Re}(z_0) = 1$, in which case f_t is not univalent (in fact $f_t(\partial\mathbb{D})$ is a slit). Thus $z_* = 1$ so, expanding $f_t(e^{i\theta})$ about $\theta = 0$, we obtain $\overline{z_0}^{-1} f_t(e^{i\theta}) - 1 = \frac{\theta^2}{1-\overline{z_0}} + \frac{i\theta^3}{(1-\overline{z_0})^2} + O(\theta^4)$, confirming that the cusp is of type $(3,2)$.

(2) Suppose that f_t continuously transitions out of univalence at $t = t_*$, so that for $t > t_*$, there exist $z \neq w \in \partial\mathbb{D}$, such that $f_t(z) - f_t(w) = 0$. Then

$$0 = \frac{f_t(z) - f_t(w)}{z - w} = -\overline{z_0}\frac{z_0 + \alpha + w(z\overline{z_0} - 1) - z}{(z\overline{z_0} - 1)(w\overline{z_0} - 1)} \qquad \Longrightarrow \qquad z_0 + \alpha + w(z\overline{z_0} - 1) - z = 0.$$

This is a linear equation in w, so it admits at most one solution. Thus $f_t(\partial\mathbb{D})$ can have at most one self intersection point, and this is a simple root, so the intersection is transversal. That is, for each $\epsilon > 0$, $f_{t_*+\epsilon}$ is not univalent in $\mathbb{D}_*$ so $f_{t_*+\epsilon}(\partial\mathbb{D}_*)$ is a topological figure-eight.

By the Whitney-Graustein theorem, which states that two regular closed curves in the plane have a regular homotopy between them if and only if they have the same degree, there is no regular homotopy between the figure eight and a simple closed curve.[19] Therefore a singularity in the derivative must form on the boundary during the transition. In particular, the derivative of f_{t_*} has a singularity at some point $z_* \in \partial\mathbb{D}$. Then, because $|z_0| \neq 1$, the form of f_t implies that $f_{t_*}'(z_*) \neq \infty$, so $f_{t_*}'(z_*) = 0$. We conclude that the only way that f_t can continuously transition out of univalence is with the formation of a $(3,2)-$cusp on $f(\partial\mathbb{D})$.

The claim (b) of the smoothness of $\partial\Omega$ for $0 < t < t_*$ then follows immediately from the fact that if a droplet in a Hele-Shaw chain contains a $(3,2)$ cusp then it is maximal ([6] §4.1). □

Univalence Criterion

For univalence of φ, it is sufficient to check that $\varphi(z) \neq \varphi(w)$ for all distinct $z, w \in \partial\mathbb{D}$. The sufficiency of injectivity on the boundary is a classic result, sometimes referred to as "Darboux's theorem":

Lemma 3.3.1 ([26], Lemma 1.1)**.** Let f be an analytic map on a Jordan domain Ω which extends continuously to the boundary. If f is injective on $\partial\Omega$ then f is univalent in Ω.

In our case, if $\varphi(z) = \varphi(w)$ but $z \neq w$, then one can show that

$$0 = \varphi(z) - \varphi(w) = (z - w)\left(c - \frac{\left(\overline{z_0} - z_0^{-1}\right)(2 - cz_0)}{(z\overline{z_0} - 1)(w\overline{z_0} - 1)} \right) \implies \frac{z\overline{z_0} - 1}{\left(|z_0|^2 - 1\right)\left(\frac{2}{cz_0} - 1\right)} = \frac{1}{w\overline{z_0} - 1}.$$

But the left and right hand sides of this equation are simply Mobius transformations, so the image of $\partial\mathbb{D}$ under each is a circle. Thus it is necessary and sufficient to check that these circles don't intersect. A pair of circles $\partial\mathbb{D}_{r_1}(x_1)$ and $\partial\mathbb{D}_{r_2}(x_2)$ intersect if and only if $(r_1 - r_2)^2 \leq |x_1 - x_2|^2 \leq (r_1 + r_2)^2$. In this case, $r_1 = |z_0|\left(|z_0|^2 - 1\right)^{-1}\left|\frac{2}{cz_0} - 1\right|^{-1}$, $x_1 = -\left(|z_0|^2 - 1\right)^{-1}\left(\frac{2}{cz_0} - 1\right)^{-1}$ and $r_2 = |z_0|\left(|z_0|^2 - 1\right)^{-1}$, $x_2 = \left(|z_0|^2 - 1\right)^{-1}$. Thus

$$|z_0|^2 \frac{(|cz_0| - |cz_0 - 2|)^2}{(|z_0|^2 - 1)^2 |cz_0 - 2|^2} \leq \frac{4}{(|z_0|^2 - 1)^2 |cz_0 - 2|^2} \leq |z_0|^2 \frac{(|cz_0| + |cz_0 - 2|)^2}{(|z_0|^2 - 1)^2 |cz_0 - 2|^2},$$

In particular, φ is univalent if and only if $|z_0| > 1$ and

$$2 < \left||cz_0| - |cz_0 - 2|\right| |z_0| \quad \text{or} \quad \left||cz_0| + |cz_0 - 2|\right| |z_0| < 2$$

However, $\left||cz_0| + |cz_0 - 2|\right| |z_0| < 2$ implies $|z_0| < 1$, and $2 < \left||cz_0| - |cz_0 - 2|\right| |z_0|$ implies $|z_0| > 1$, so we may conclude that φ is univalent if and only if

$$2 < \left||cz_0| - |cz_0 - 2|\right| |z_0|. \tag{3.14}$$

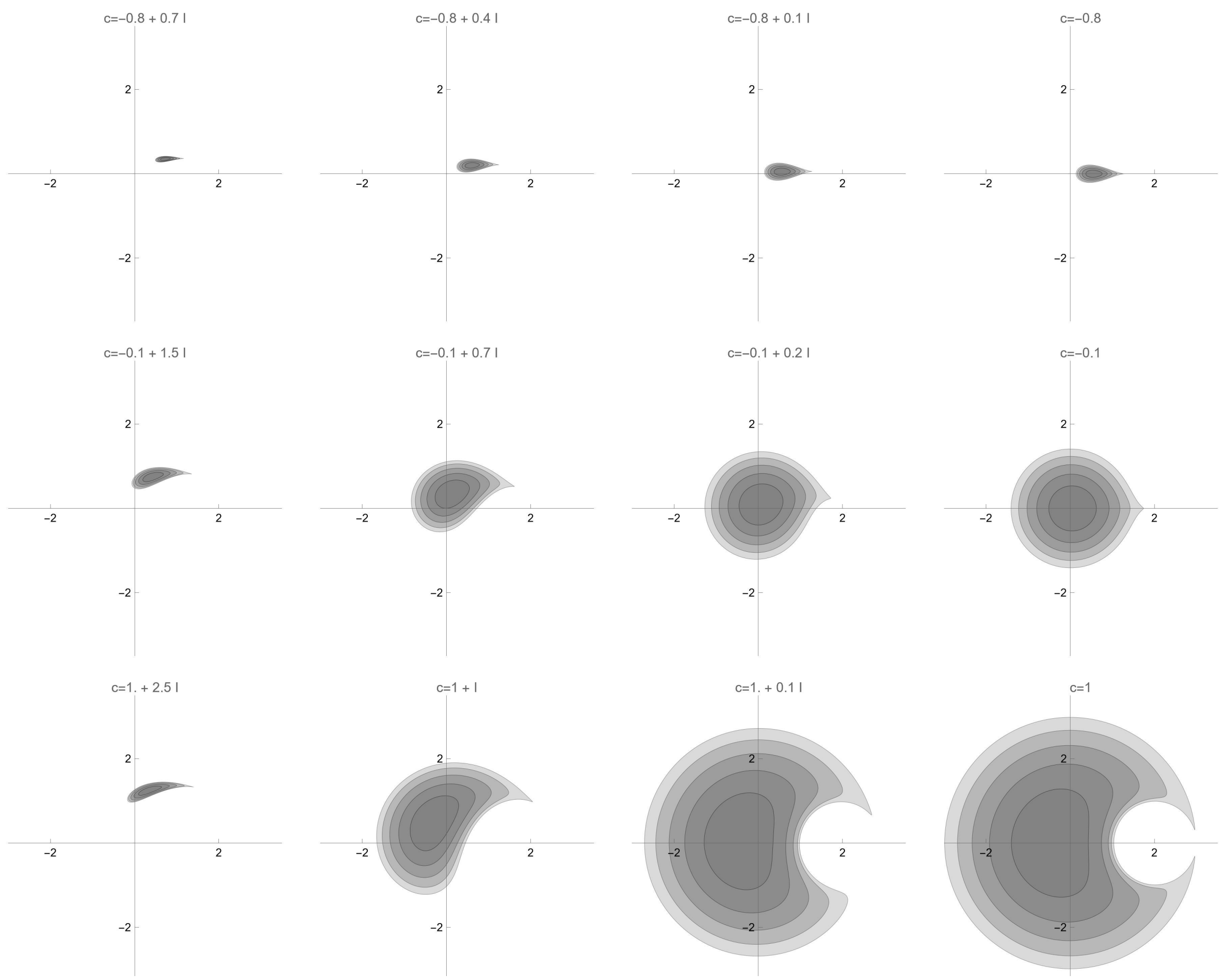

Figure 3.4: $\Omega_t \in \mathrm{QD}\left(\frac{\alpha}{w-2}\right)$ (complements of shaded) for various values of α, and increasing t up to the critical value. Note that the critical value is decreasing as $|w_0|^2 + 2\mathrm{Re}(\alpha) - 2|\alpha|$ approaches 0.

Chapter 4

POWER-WEIGHTED QUADRATURE DOMAINS

We will begin our study of weighted quadrature domains by discussing the instructive example of *power-weighted quadrature domains* (PQDs). Informally, a PQD is a domain $\Omega \subset \widehat{\mathbb{C}}$ if it admits a quadrature identity in which the integral over the domain is taken with respect to the weight $\rho_a(w) = |w|^{2(a-1)}$ for some $a > 0$. The choice of ρ_a is motivated by the correspondence between PQDs and local droplets of potentials of the form

$$Q(w) = \frac{|w|^{2a}}{a^2} - 2\mathrm{Re}(H(w)).$$

The exact nature of this correspondence is detailed in Propositions 4.1.1, 4.4.1, and 4.4.2. Before a deeper study of PQDs, we begin with a brief discussion of weighted quadrature domains, which are the subject of §6.

4.1 Weighted Quadrature Domains

As the name suggests, a $\rho-$weighted quadrature domain is a domain $\Omega \subset \widehat{\mathbb{C}}$ satisfying a quadrature identity in which the area integral is taken with respect to the weight $\rho \geq 0$. Consider $\rho(w) = |w|^4 = \frac{1}{4}\Delta\frac{|w|^6}{9}$ as a first example.

Example 4.1.1. If $c > 0$, then the unbounded domain $\Omega_c = \{w \in \widehat{\mathbb{C}} : |w^3 - 1| > c^3\}$ is a WQD with respect to the weight $\rho(w) = |w|^4$. To see this, note that $|w^3 - 1|^2 \doteq c^6$, so $\overline{w}^3 \doteq \frac{c^6}{w^3-1} + 1$. Thus for each $f \in \mathcal{A}_0(\Omega)$,

$$\begin{aligned}\int_{\Omega} f(w)|w|^4 dA(w) &= \oint_{\partial\Omega} f(w)\frac{w^2}{3}\overline{w}^3 dw \\ &= \oint_{\partial\Omega} f(w)\frac{1}{3}w^2\left(\frac{c^6}{w^3-1} + 1\right) dw\end{aligned}$$

Applying the residue theorem and the fact that the third roots of unity are not in Ω, we find

$$\int_{\Omega} f(w)|w|^4 dA(w) = -\frac{f_3}{3},$$

where $f(w) = f_1 w^{-1} + f_2 w^{-2} + f_3 w^{-3} + O(w^{-3})$.

We observe in Figure 4.1 that the boundary of Ω forms corners at the origin. Here, the angle is $\frac{\pi}{3}$. This is a special case of a more general phenomenon: the angle of intersection of the boundary of a PQD with the origin is observed to be an integer multiple of $\frac{\pi}{a}$ whenever $a > \frac{1}{2}$.

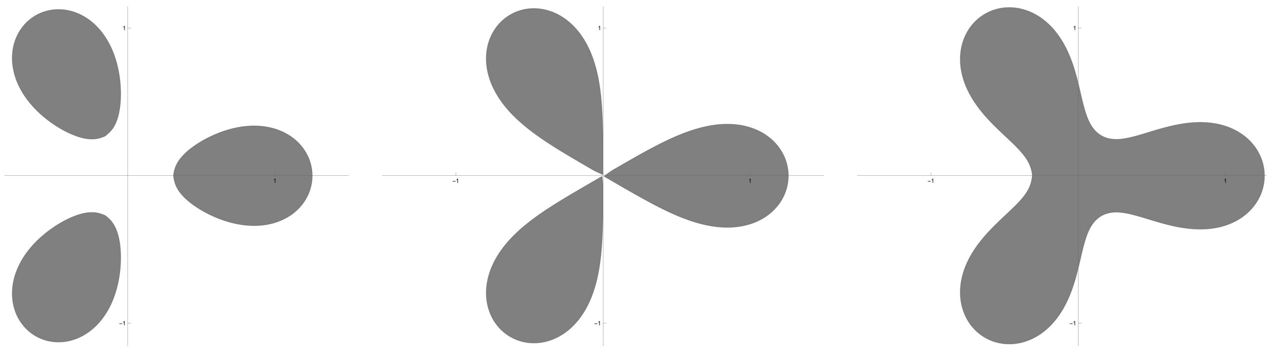

Figure 4.1: $\Omega \in \mathrm{QD}_{|w|^4}\left(\frac{w^2}{3}; \mathcal{A}_0(\Omega)\right)$ (the complements of the shaded regions) for $c = .95, 1, 1.05$ transitioning from multiply connected to simply connected (left to right)

As for classical quadrature domains, the existence of a *weighted quadrature identity* is equivalent to the existence of a rational function h for which Equation 4.1 holds over some analytic test class $\mathcal{F}$ (e.g $h(w) = \frac{w^2}{3}$ in example 4.1.1). This motivates the following definition.

Definition 4.1.1 (Weighted quadrature domain)**.** Let $\Omega \subset \widehat{\mathbb{C}}$ be a rectifiable domain and ρ an *admissible* weight function on Ω. Ω is a $\rho-$weighted quadrature domain with respect to the $\rho-$integrable analytic test class $\mathcal{F}$ if there exists a rational function h for which

$$\int_{\Omega} f \rho dA = \oint_{\partial\Omega} f(w)h(w)dw \tag{4.1}$$

holds for all $f \in \mathcal{F}$. This is denoted by $\Omega \in \mathrm{QD}_{\rho}(h; \mathcal{F})$.

A weight ρ on Ω is said to be *admissible* if it is measurable, non-negative, and satisfies $0 < \rho < \infty$ almost everywhere. The associated measure

$$\nu(U) := \int_U \rho dA$$

is then positive, $\sigma-$finite, and absolutely continuous with respect to the Lebesgue measure dA, with Radon-Nikodym derivative $\frac{d\nu}{dA} = \rho$.

While this definition is very general, one can still recover analogs of many of the results for classical QDs - at least in certain special cases. The following proposition provides an analogue of the coincidence equation (3.3) for a broad class of radially symmetric weights.

Proposition 4.1.1. Let $\Omega \in \mathrm{QD}_{\rho}(h; \mathcal{F})$ be a bounded domain where $\rho = \frac{\partial^2 F}{\partial w \partial \overline{w}} = \frac{1}{4}\Delta F$ for some real valued $F \in C^2(\mathrm{Cl}(\Omega))$, radially symmetric with respect to the origin. If $\mathcal{A}(\Omega) \subseteq \mathcal{F}$, then there exists $G \in \mathcal{A}(\Omega)$ such that

$$\frac{\partial F}{\partial w} \doteq h(w) + G(w). \tag{4.2}$$

The same holds when Ω is unbounded, with each $\mathcal{A}(\Omega)$ above replaced by $\mathcal{A}_0(\Omega)$.

Proof of Proposition 4.1.1. Let $\Omega \in \mathrm{QD}_\rho(h;\mathcal{F})$ be a bounded domain with $\rho = \frac{1}{4}\Delta F$, where $F \in C^2(\mathrm{Cl}(\Omega))$ is radially symmetric with respect to the origin. By assumption if $f \in \mathcal{A}(\Omega)$ then $f \in \mathcal{F}$, so applying the quadrature identity and Green's theorem, we obtain

$$\begin{aligned}\oint_{\partial\Omega} f(w)\left(h(w) - \frac{\partial F}{\partial w}\right)dw &= \oint_{\partial\Omega} f(w)h(w)dw - \oint_{\partial\Omega} f(w)\frac{\partial F}{\partial w}dw \\ &= \int_{\Omega} f\rho dA - \int_{\Omega} f\rho dA = 0\end{aligned}$$

Therefore there exists $G \in \mathcal{A}(\Omega)$ such that $\frac{\partial F}{\partial w} \doteq h + G$. The proof in the unbounded case is entirely analogous. $\square$

This generalized coincidence equation naturally leads to the notion of a *generalized Schwarz function* associated to F and Ω, $\widetilde{S} := h + G \doteq \frac{\partial F}{\partial w} \in \mathcal{M}(\Omega)$. For example if Ω is a bounded domain in $\mathrm{QD}_{|w|^{2(a-1)}}(h;\mathcal{A}(\Omega))$, so that $F(w) = \frac{|w|^{2a}}{a^2}$ for some $a > 0$, then Proposition 4.1.1 implies

$$\widetilde{S}(w) := h(w) + G(w) \doteq \frac{1}{a}\overline{w}|w|^{2(a-1)},$$

for some $G \in \mathcal{A}(\Omega)$ (see Equation 4.3). In the remainder of this section, we explore various families of PQDs and their properties.

4.2 Definition and Basic Properties

Power-weighted quadrature domains behave very much like classical QDs away from the origin. However, the singularity of $\rho_a(w) = |w|^{2(a-1)}$ at the origin gives rise to behavior which is excluded in the classical case (by Sakai regularity, e.g. Theorem 3.2 of [20]): the development of "corners" on the boundary (e.g. in Figures 4.1, 4.2, 4.3, and 4.4). It appears that this phenomenon isn't unique to PQDs, but is generic among WQDs with a singular weight.

As in the classical case, the choice of test class $\mathcal{F}$ is of great importance when it comes to the uniqueness of the quadrature function. For example if Ω were unbounded then for each $\alpha \in \mathbb{C}$,

$$\oint_{\partial\Omega} f(w)\,(\alpha + h(w))\,dw = \oint_{\partial\Omega} f(w)h(w)dw$$

whenever $f \in L^1_a(\Omega;\rho_a)$ and $a \geq 1$. That is, $\mathrm{QD}_{\rho_a}(h; L^1_a(\Omega;\rho_a)) = \mathrm{QD}_{\rho_a}(h+\alpha; L^1_a(\Omega;\rho_a))$ for all $\alpha \in \mathbb{C}$. This isn't the case, however, when $\mathcal{F} = \mathcal{A}_0(\Omega)$ because if $w \in \Omega_*$ then $\xi \mapsto (\xi - w)^{-1} \in \mathcal{A}_0(\Omega)$, so

$$\oint_{\partial\Omega} \frac{\alpha + h(\xi)}{\xi - w}d\xi = \alpha + \oint_{\partial\Omega} \frac{h(\xi)}{\xi - w}d\xi \neq \oint_{\partial\Omega} \frac{h(\xi)}{\xi - w}d\xi.$$

Applying Proposition 4.1.1 with $F(w) = \frac{|w|^{2a}}{a^2}$, we find that there exists $G \in \mathcal{A}(\Omega)$ ($\mathcal{A}_0(\Omega)$ when Ω is unbounded) for which

$$\frac{1}{a}\overline{w}|w|^{2(a-1)} \doteq h(w) + G(w). \tag{4.3}$$

Note this implies that when Ω is bounded (resp. unbounded), h is uniquely associated to Ω if we restrict to the class $\mathrm{Rat}_0(\Omega)$ (resp. $\mathrm{Rat}(\Omega)$): If Ω is a PQD with $\mathcal{F} = \mathcal{A}(\Omega)$ (resp. $\mathcal{A}_0(\Omega)$) two such quadrature functions h_1, h_2 then

$$h_1(w) - h_2(w) \doteq \left(\frac{1}{a}\overline{w}|w|^{2(a-1)} - G_1(w)\right) - \left(\frac{1}{a}\overline{w}|w|^{2(a-1)} - G_2(w)\right) = G_2(w) - G_1(w).$$

Hence $h_1 - h_2$ continues analytically into Ω. So when Ω is bounded, $h_1 - h_2 \in \mathrm{Rat}_0(\Omega) \cap \mathcal{A}(\Omega) = \{0\}$, and when Ω is unbounded, $h_1 - h_2 \in \mathrm{Rat}(\Omega) \cap \mathcal{A}_0(\Omega) = \{0\}$, so $h_1 = h_2$. This motivates the following definition.

Definition 4.2.1 (Power-weighted quadrature domain)**.** Take $a > 0$ and let Ω be a bounded (resp. unbounded) rectifiable domain equal to the interior of its closure. We say that Ω is a PQD if there exists $h \in \mathrm{Rat}_0(\Omega)$ (resp. $\mathrm{Rat}(\Omega)$) for which

$$\int_\Omega f \rho_a dA = \oint_{\partial\Omega} f(w)h(w)dw \tag{4.4}$$

is satisfied for all $f \in \mathcal{A}(\Omega)$ (resp. $A_0(\Omega)$). This is denoted by $\Omega \in \mathrm{QD}_a(h)$.

In the language of general weighted quadrature domains, this is equivalent to the statement that $\Omega \in \mathrm{QD}_{\rho_a}(h; \mathcal{A}(\Omega))$ with $h \in \mathrm{Rat}_0(\Omega)$ when Ω is bounded, and $\Omega \in \mathrm{QD}_{\rho_a}(h; \mathcal{A}_0(\Omega))$, $h \in \mathrm{Rat}(\Omega)$ when Ω is unbounded. We also denote by $\Omega \in \mathrm{QD}_a$ that there exists an h for which $\Omega \in \mathrm{QD}_a(h)$. There are several equivalent characterizations of PQDs in the spirit of Proposition 3.1.1:

Theorem 4.2.1. *If $\Omega \subset \widehat{\mathbb{C}}$ is a rectifiable domain equal to the interior of its closure and $a > 0$ then the following are equivalent*

1. *There exists a rational h for which $\Omega \in QD_a(h)$.*
2. *There exists a $G \in \mathcal{A}(\Omega)$ ($\mathcal{A}_0$ when Ω is unbounded) and a rational h for which*

$$\frac{1}{a}\overline{w}|w|^{2(a-1)} \doteq h(w) + G(w).$$

3. *There exists a function $S_a \in \mathcal{M}(\Omega)$ for which $S_a(w) \doteq \frac{1}{a}\overline{w}|w|^{2(a-1)}$. This is the* generalized Schwarz function *associated to Ω.*

Moreover in this case $h = C_{\Omega_*}[S_a]$, and G is the ρ_a−weighted Cauchy transform of Ω_*,

$$G(w) = \int_{\Omega_*} \frac{|\xi|^{2(a-1)}}{w - \xi} dA(\xi). \tag{4.5}$$

The generalized Schwarz function $S_a \in \mathcal{M}(\Omega)$ is unique when it exists as a consequence of its equality with $\frac{1}{a}\overline{w}|w|^{2(a-1)}$ on $\partial\Omega$. We'll see in the next section that simply connected PQDs admit an additional characterization in terms of their Riemann map.

Proof of Theorem 4.2.1.

(1) $\implies$ **(2)**: This is immediate from Equation 4.3 (via Proposition 4.1.1).

(2) $\implies$ **(3)**: $\frac{1}{a}\overline{w}|w|^{2(a-1)} \doteq h(w) + G(w) =: S_a(w)$, is the generalized Schwarz function for Ω because $h + G \in \mathcal{M}(\Omega)$. Also, $C_{\Omega_*}[S_a] = C_{\Omega_*}[h + G] = h$.

(3) $\implies$ **(1)**: Finally, if Ω is a bounded domain for which such an S_a exists, and $f \in \mathcal{A}(\Omega)$, then

$$\begin{aligned}\int_\Omega f(w)|w|^{2(a-1)}dA(w) &= \oint_{\partial\Omega} f(w)\frac{1}{a}\overline{w}|w|^{2(a-1)}dw = \oint_{\partial\Omega} f(w)S_a(w)dw\\ &= \oint_{\partial\Omega} f(w)C_{\Omega_*}[S_a(w)]\,dw = \oint_{\partial\Omega} f(w)h(w)dw,\end{aligned}$$

So $\Omega \in \mathrm{QD}_a(h)$. The argument in the unbounded case is completely analogous.

Finally, we verify Equation 4.5: First, if Ω is unbounded then for each $w \in \Omega$,

$$G(w) = \oint_{\partial\Omega}\frac{G(\xi)d\xi}{\xi - w} = -\oint_{\partial\Omega_*}\frac{\frac{1}{a}\overline{\xi}|\xi|^{2(a-1)} - h(\xi)}{\xi - w}d\xi = \oint_{\partial\Omega_*}\frac{\frac{1}{a}\overline{\xi}|\xi|^{2(a-1)}}{w - \xi}d\xi = \int_{\Omega_*}\frac{|\xi|^{2(a-1)}}{w - \xi}dA(\xi).$$

The first equality is from Cauchy's integral formula, the second follows from (2), the third follows from the fact that $h \in \mathcal{A}(\Omega_*)$, and the final equality is Green's formula. On the other hand, if Ω is bounded then Ω_* is unbounded, so we consider the integral in terms of its Cauchy principal value. In particular, we have by Green's theorem that for each $w \in \Omega$,

$$\int_{\Omega_*}\frac{|\xi|^{2(a-1)}}{\xi - w}dA(\xi) := \lim_{r\to\infty}\int_{\Omega_*\cap\mathbb{D}_r}\frac{|\xi|^{2(a-1)}}{\xi - w}dA(\xi) = \lim_{r\to\infty}\oint_{\partial(\Omega_*\cap\mathbb{D}_r)}\frac{\frac{1}{a}\overline{\xi}|\xi|^{2(a-1)}}{\xi - w}d\xi$$

Integrating over each component of the boundary separately, and applying the fact that

$$\frac{1}{a}\overline{\xi}|\xi|^{2(a-1)} = h(\xi) + G(\xi)$$

on $\partial\Omega$, we find

$$\int_{\Omega_*}\frac{|\xi|^{2(a-1)}}{\xi - w}dA(\xi) = \oint_{\partial\Omega_*}\frac{h(\xi) + G(\xi)}{\xi - w}d\xi + \lim_{r\to\infty}\frac{r^{2a}}{a}\oint_{\partial\mathbb{D}_r}\frac{\xi^{-1}}{\xi - w}d\xi.$$

Applying Cauchy's theorem and that $h \in \mathcal{A}_0(\Omega)$, we obtain

$$\int_{\Omega_*}\frac{|\xi|^{2(a-1)}}{\xi - w}dA(\xi) = -\oint_{\partial\Omega}\frac{G(\xi)}{\xi - w}d\xi + \lim_{r\to\infty}\frac{r^{2a}}{a}\cdot 0 = -G(w).$$

$\square$

Boundary Regularity of PQDs

Recall that the boundary regularity of $\Omega \in \mathrm{QD}$ (Corollary 3.1.1) was a direct consequence of the existence of an associated Schwarz function and Sakai's regularity theorem 3.1.1, which characterizes the types of singularities which can occur on a curve admitting a local Schwarz function. Unfortunately, Sakai's result can't be applied directly to PQDs, as they don't generally admit a Schwarz function. However, a recent result of [32] (Corollary 2.5) sufficiently generalizes Sakai's result for our purposes.

Lemma 4.2.1. Let Ω be a bounded domain, p a polynomial, F a function analytic in a neighborhood of $\mathrm{Cl}(\Omega)$, and $T \in \mathcal{A}(\Omega)$. Suppose that for all $w \in \partial\Omega$, we have

$$T(w) = p(\overline{w})F(w).$$

Then for each $w_0 \in \partial\Omega$ such that $p'(\overline{w_0}) \neq 0$, there exists $\delta > 0$ such that $p^{-1}\left(\frac{T}{F}\right)$ is a local Schwarz function at w_0.

In particular, if $a \in \mathbb{Z}_+$ and $\Omega \in \mathrm{QD}_a(h)$ is a bounded PQD, Theorem 4.2.1 tells us that there exists a $G \in \mathcal{A}(\Omega)$ such that $a(h(w) + G(w)) = \overline{w}|w|^{2(a-1)}$. If we let $r(w) = \prod_k (w - a_k)^{n_k}$ (where the a_k are the poles of h and n_k their respective orders), then

$$p(\overline{w})F(w) := \overline{w}^a (w^{a-1} r(w)) \doteq ar(w)(h(w) + G(w)) =: T(w),$$

where T is analytic in Ω, $F(w) := w^{a-1}r(w)$ is analytic in a neighborhood of $\mathrm{Cl}(\Omega)$, and $p(w) = w^{a-1}$ is a polynomial. This choice of T, p, and F satisfy the conditions of Lemma 4.2.1, and we find that Ω admits a local Schwarz function at every point $w_0 \in \partial\Omega$, except possibly at $w_0 = 0$. We conclude that if Ω is a bounded PQD then for each $\epsilon > 0$, $\partial\Omega \setminus \mathbb{D}_\epsilon$ has finitely many singular points, each of which is either a cusp or a double point. On the other hand, if Ω is an unbounded PQD, then choose $w_1 \notin \mathrm{Cl}(\Omega)$ and consider the bounded domain $\widetilde{\Omega} = (\Omega - w_1)^{-1}$, so $\partial\widetilde{\Omega}$ satisfies

$$\frac{1}{h((w + w_1)^{-1}) + G((w + w_1)^{-1})} = a\overline{(w + w_1)}|w + w_1|^{2(a-1)}$$

for some $G \in \mathcal{A}_0(\Omega)$. If we let $r(w) = \prod_k (w - a_k)^{n_k}$ (where the a_k are the poles of the left hand side and n_k their respective orders), then

$$p(\overline{w})F(w) := a(\overline{w} + \overline{w_1})|w + w_1|^{2(a-1)} r(w) \doteq \frac{r(w)}{h((w + w_1)^{-1}) + G((w + w_1)^{-1})} =: T(w),$$

where T is analytic in $\widetilde{\Omega}$, $F(w) := ((w + w_1)^{a-1} r(w))$ is analytic in a neighborhood of $\mathrm{Cl}(\widetilde{\Omega})$, and $p(w) = a(w + \overline{w_1})^a$ is a polynomial. This choice of T, p, and F satisfy the conditions of Lemma 4.2.1, and we find that $\widetilde{\Omega}$ admits a local Schwarz function at every point of $\partial\widetilde{\Omega}$, except possibly at $-w_1$. The map $w \mapsto (w - w_1)^{-1}$ is univalent in a neighborhood of $\partial\Omega$, so this implies that Ω admits a local Schwarz function at every point of $\partial\Omega$, except possibly at 0. In summary,

Corollary 4.2.1. If $\Omega \in \mathrm{QD}_a$ for some $a > 0$ then, for each $\epsilon > 0$, $\partial\Omega \setminus \mathbb{D}_\epsilon$ has finitely many singular points, each of which is either a cusp or a double point.

To constrain the behavior at the origin, we require an extra ingredient, that $(\partial\Omega)^a$ is a subset of an algebraic curve:

Theorem 4.2.2. *If $\Omega \in QD_a$ for some $a \in \mathbb{Z}_+$, there exists a non-constant polynomial p, irreducible over $\mathbb{C}$, with real coefficients such that $p(x, y) = 0$ for all $x + iy \in \partial\Omega^a$. (c.f. [1], Theorem 3). In particular, $\partial\Omega$ is a subset of the ath root of an algebraic curve.*

Proof of Theorem 4.2.2. By Theorem 4.2.1, Ω admits a generalized Schwarz function, S_a which is analytic in Ω except at finitely many poles such that $T(w) := \frac{S_a(w)}{\frac{1}{a}w^{a-1}} \doteq \overline{w}^a$. Thus

$$f(w) := \frac{1}{2}(w^a + T(w)), \text{ and } g(w) := \frac{1}{2i}(w^a - T(w))$$

are analytic except at finitely many poles, and admit continuous real extensions to $\partial\Omega$, $f(w) \doteq \mathrm{Re}(w^a)$, $g(w) \doteq \mathrm{Im}(w^a)$. Now suppose Ω is bounded. We apply Theorem 2 of [1]:

Let $\Omega \subset \mathbb{C}$ be a bounded domain and suppose f and g are holomorphic in Ω except at finitely many poles. Further suppose that f and g admit real, continuous, extensions to $\partial\Omega$. Then there exists a non-trivial polynomial $p(x, y)$, irreducible over $\mathbb{C}$, with real coefficients such that $p(f(z), g(z)) = 0$ on $\partial\Omega$.

Therefore there exists a real non-constant, $\mathbb{C}$-irreducible polynomial p such that $p(f(w), g(w)) = 0$ on $\partial\Omega$. Thus, $p(\mathrm{Re}(w^a), \mathrm{Im}(w^a)) = 0$ on $\partial\Omega$. This is equivalent to $p(x, y) = 0$ for all $x + iy \in \partial\Omega^a$. The argument is similar, but slightly more delicate in the unbounded case; see the proof of Theorem 3 in [1]. □

Theorem 4.2.2 straightforwardly implies that $\partial\Omega$ has finitely many singular points when a is a positive integer. We conclude:

Theorem 4.2.3. *If $\Omega \in QD_a$ for some $a \in \mathbb{Z}_+$ then $\partial\Omega$ has finitely many singular points, where each non-zero singular point is either a cusp or a double point.*

4.3 Simply Connected PQDs

We are now prepared to consider two different generalizations of the Riemann mapping characterization and formulae for simply connected classical QDs (Theorems 3.2.2 and 3.2.3) to PQDs for arbitrary $a > 0$. In particular, we exhibit Faber transform formulae in terms of both sum (Mittag-Leffler) and product (inner-outer/Nevanlinna) decompositions of the Riemann map φ.

Each of these approaches come with their own benefits and drawbacks. The sum representation yields relatively simple formulae, but only applies for integer values of a. On the other hand, the product representation yields more specific and unwieldy formulae, but applies for all $a > 0$. We will begin with a discussion of the simpler sum representation.

General Faber Sum Identities for PQDs

Fix $a \in \mathbb{Z}_+$ and let Ω be a simply connected unbounded PQD with quadrature function h and Riemann map $\varphi : \mathbb{D}_* \to \Omega$. Note that $\varphi^a(z)$ can be rewritten as $p + g$, where $p \in \mathcal{A}(\mathbb{D})$ is a polynomial of degree a and $g \in \mathcal{A}_0(\mathbb{D}_*)$. Recalling the definition of Faber polynomials (Equation 2.9), we find that p is, in fact, the ath inverse Faber polynomial W_a,

$$p = C_{\mathbb{D}}\left[p\right] = C_{\mathbb{D}}\left[\varphi^a - g\right] = C_{\mathbb{D}}\left[\varphi^a\right] = W_a.$$

As a consequence, $C_{\mathbb{D}}\left[(\varphi^a)^\#\right] = (\varphi^a)^\# - \mathring{W}_a^{\#} \in \mathcal{A}(\mathbb{D})$, where $\mathring{W}_a := W_a - W_a(0)$, so we may compute its Faber transform:

$$\Phi_\varphi\left((\varphi^a)^\# - \mathring{W}_a^{\#}\right) = \Phi_\varphi\left(C_{\mathbb{D}}\left[(\varphi^a)^\#\right]\right).$$

Note that $(\varphi^a)^\# \doteq \overline{\varphi^a}$, so substituting Equation 4.3, we obtain

$$(\varphi^a)^\# \doteq \left(\frac{h(w) + G(w)}{\frac{1}{a}w^{a-1}}\right) \circ \varphi,$$

for some $G \in \mathcal{A}_0(\Omega)$. Then, as $C_{\mathbb{D}}$ and the Faber transform only respect the boundary values of their argument, we may substitute to obtain

$$\Phi_\varphi\left((\varphi^a)^\# - \mathring{W}_a^{\#}\right) = \Phi_\varphi\left(C_{\mathbb{D}}\left[\left(\frac{h(w) + G(w)}{\frac{1}{a}w^{a-1}}\right) \circ \varphi\right]\right).$$

We recognize the argument of the RHS as the inverse Faber transform. Applying Equation 2.7 yields

$$\begin{aligned}\Phi_\varphi\left((\varphi^a)^\# - \mathring{W}_a^{\#}\right)(w) &= C_{\Omega_*}\left[\left(\frac{h(w) + G(w)}{\frac{1}{a}w^{a-1}}\right) \circ \varphi \circ \psi(w)\right] \\ &= C_{\Omega_*}\left[\frac{h(w) + G(w)}{\frac{1}{a}w^{a-1}}\right].\end{aligned}$$

Applying the inverse Faber transform to both sides of the above equation and rearranging, we obtain

$$\varphi^a(z) = \mathring{W}_a(z) + \Phi_\varphi^{-1}\left(C_{\Omega_*}\left[\frac{h(w) + G(w)}{\frac{1}{a}w^{a-1}}\right]\right)^\#(z). \tag{4.6}$$

Furthermore, the Faber transform of a meromorphic function is rational, so we may conclude that $\varphi^a \in \mathrm{Rat}(\mathbb{D})$. This proves the backward direction of the first of the two main theorems of this section:

Theorem 4.3.1. *Take $a \in \mathbb{Z}_+$ and let Ω be an unbounded simply connected domain. Then there exists $r \in \mathrm{Rat}(\mathbb{D}_*)$ for which $\varphi^a(z) = W_a(z) - W_a(0) + r^\#(z)$ is a Riemann map for Ω if and only if $\Omega \in QD_a(h)$ for some $h \in \mathrm{Rat}(\Omega)$.*

Theorem 4.3.2. *Take $a \in \mathbb{Z}_+$ and let Ω be a bounded simply connected domain. Then there exists $r \in \mathrm{Rat}_0(\mathbb{D})$ for which $\varphi^a(z) = \varphi^a(0) + r^\#(z)$ is a Riemann map for Ω if and only if $\Omega \in QD_a(h)$ for some $h \in \mathrm{Rat}_0(\Omega)$.*

In the course of the proof of Theorem 4.3.2, we also obtain the following pair of equations relating h and φ.

Corollary 4.3.1. Whenever r and h, as in Theorems 4.3.1 and 4.3.2, exist, they satisfy the following pair of identities

$$r(z) = \Phi_\varphi^{-1}\left(\mathcal{C}_{\Omega_*}\left[\frac{h(w) + G(w)}{\frac{1}{a}w^{a-1}}\right]\right)(z), \qquad h(w) = \mathcal{C}_{\Omega_*}\left[\frac{1}{a}w^{a-1}(\varphi^a)^\# \circ \psi(w)\right]. \tag{4.7}$$

Where $G \in \mathcal{A}(\Omega)$ when Ω is bounded and $G \in \mathcal{A}_0(\Omega)$ when Ω is unbounded.

Note that the $G(w)$ term in the left equation drops out whenever $0 \notin \Omega$. We now prove Theorem 4.3.2 and Corollary 4.3.1. The forward direction of the proof of Theorem 4.3.1 will be omitted, as it is analogous.

Proof of Theorem 4.3.2 and Corollary 4.3.1.
For the forward direction, suppose there exists $r \in \mathrm{Rat}_0(\mathbb{D})$ for which $\varphi^a(z) = \varphi^a(0) + r^\#(z)$. Then note that for $w \in \partial\Omega$,

$$\begin{aligned}\frac{1}{a}\overline{w}|w|^{2(a-1)} = \frac{1}{a}w^{a-1}\overline{w}^a &= \frac{1}{a}w^{a-1}\overline{\varphi^a \circ \psi(w)} \\ &= \frac{1}{a}w^{a-1}\overline{\left(\varphi^a(0) + r^\# \circ \psi(w)\right)} \\ &= \frac{1}{a}w^{a-1}\left(\overline{\varphi^a(0)} + r \circ \psi(w)\right) =: S_a(w)\end{aligned}$$

extends meromorphically to Ω, so S_a is a generalized Schwarz function for Ω. Thus $\Omega \in \mathrm{QD}_a(h)$ for some $h \in \mathrm{Rat}_0(\Omega)$ by Theorem 4.2.1.

For the reverse direction, suppose that $\Omega \in \mathrm{QD}_a(h)$ for some $h \in \mathrm{Rat}_0(\Omega)$. Then, as $\mathcal{C}_{\mathbb{D}_*}\left[\overline{\varphi^a(0)}\right] = 0$ and $(\varphi^a)^\# - \overline{\varphi^a(0)} \in \mathcal{A}_0(\mathbb{D}_*)$, we find that $\mathcal{C}_{\mathbb{D}_*}\left[(\varphi^a)^\#\right] = (\varphi^a)^\# - \overline{\varphi^a(0)}$, so

$$\Phi_\varphi\left((\varphi^a)^\# - \overline{\varphi^a(0)}\right) = \Phi_\varphi\left(\mathcal{C}_{\mathbb{D}_*}\left[(\varphi^a)^\#\right]\right).$$

Rearranging Equation 4.3 and substituting as we did in the bounded case, we obtain

$$\Phi_\varphi\left((\varphi^a)^\# - \overline{\varphi^a(0)}\right) = \Phi_\varphi\left(\mathcal{C}_{\mathbb{D}_*}\left[\left(\frac{h(w)+G(w)}{\frac{1}{a}w^{a-1}}\right)\circ\varphi\right]\right).$$

We recognize the argument of the RHS as the inverse Faber transform. Applying Equation 2.4 yields

$$\Phi_\varphi\left((\varphi^a)^\# - \overline{\varphi^a(0)}\right)(w) = \mathcal{C}_{\Omega_*}\left[\left(\frac{h(w)+G(w)}{\frac{1}{a}w^{a-1}}\right)\circ\varphi\circ\psi(w)\right] = \mathcal{C}_{\Omega_*}\left[\frac{h(w)+G(w)}{\frac{1}{a}w^{a-1}}\right].$$

Taking the inverse Faber transform of both sides and noting that the RHS is meromorphic in Ω, so that its inverse Faber transform is rational, we obtain the desired result.

Finally, we verify the formula for the quadrature function. By Theorem 4.2.1, $\frac{1}{a}w^{a-1}\overline{w}^a \doteq h(w) + G(w)$ for some $G \in \mathcal{A}(\Omega)$, so

$$h(w) = \mathcal{C}_{\Omega^*}\left[h(w)\right] = \mathcal{C}_{\Omega^*}\left[\frac{1}{a}w^{a-1}\overline{w}^a - G(w)\right] = \mathcal{C}_{\Omega^*}\left[\frac{1}{a}w^{a-1}\overline{w}^a\right],$$

where the first equality follows from the fact that $h \in \mathcal{A}_0(\Omega_*)$ and the last from the fact that $G \in \mathcal{A}(\Omega)$. Thus, noting that $\overline{w}^a \doteq (\varphi^a)^\# \circ \psi(w)$ we find that $h(w) = \mathcal{C}_{\Omega^*}\left[\frac{1}{a}w^{a-1}(\varphi^a)^\# \circ \psi(w)\right]$. □

As a demonstration of the utility of this approach, we consider a simple one-point PQD for $a = 2$.

Non-Singular One-Point PQDs with a=2

Proposition 4.3.1. If $\alpha > 0$ and $\Omega \in \mathrm{QD}_2\left(\frac{\alpha}{w-1}\right)$ is a simply connected domain not containing 0, then Ω is the unique connected component of the square root of $\mathbb{D}_{2\sqrt{\alpha}}(1)$ containing 1.

Proof of Proposition 4.3.1. Let $0 \notin \Omega \in \mathrm{QD}_2(h)$ be a simply connected bounded domain with $h(w) = \frac{\alpha}{w-1}$ for some $\alpha > 0$. Furthermore, let $\varphi : \mathbb{D} \to \Omega$ be the unique Riemann map associated to Ω for which $\varphi(0) = 1$ and $\varphi'(0) > 0$. Then by Theorem 4.3.2 and corollary 4.3.1, there exists $r \in \mathrm{Rat}_0(\mathbb{D})$ for which $\varphi^2 = 1 + r^\#$ is a Riemann map for Ω and r is given by Equation 4.7 for some

$G \in \mathcal{A}(\Omega)$. Thus,

$$\begin{aligned} r(z) &= \Phi_\varphi^{-1}\left(C_{\Omega_*}\left[2\frac{\frac{\alpha}{w-1}+G(w)}{w}\right]\right)(z) = 2\Phi_\varphi^{-1}\left(C_{\Omega_*}\left[\frac{\alpha}{w-1}-\frac{\alpha-G(w)}{w}\right]\right)(z) \\ &= 2\alpha\Phi_\varphi^{-1}\left(\frac{1}{w-1}\right)(z) = \frac{2\alpha\psi'(1)}{z-\psi(1)}, \end{aligned}$$

where the last equality follows from Equation 2.9. Also $\psi(1) = \varphi^{-1}(1) = 0$ and $\psi'(1) = \varphi'(0)^{-1}$, so $r(z) = \frac{2\alpha}{\varphi'(0)z}$, and we find $\varphi^2(z) = 1 + \frac{2\alpha}{\varphi'(0)}z$. Thus

$$2\varphi'(0) = 2\varphi(0)\varphi'(0) = (\varphi^2)'(0) = \frac{2\alpha}{\varphi'(0)},$$

so $\varphi'(0) = \sqrt{\alpha}$, and we conclude

$$\varphi^2(z) = 1 + 2\sqrt{\alpha}z.$$

□

In §4.4 we provide a classification of simply connected one-point PQDs more generally.

Non-Singular Linear PQDs with a=2

As a second example application of the Faber transform sum formulae, we classify simple connected $2-$PQDs with a linear quadrature function.

Proposition 4.3.2. Fix $\alpha_0 \in \mathbb{C}$, $\alpha_1 \in \mathbb{C} \setminus \{0\}$, and a simply connected domain $0 \notin \Omega \in \mathrm{QD}_2(\alpha_0 + \alpha_1 w)$. Then Ω is unique modulo conformal radius unless $\alpha_0 = 0$, in which case $\pm\Omega$ are the only such domains.

Proof of Proposition 4.3.2. Let Ω be a simply connected domain not containing zero such that $\Omega \in \mathrm{QD}_2(h)$, where $h(w) = \alpha_0 + w$ for some $\alpha_0 \in \mathbb{C}$ (by change of variables, we can wlog assume $\alpha_1 = 1$ as long as $\alpha_1 \neq 0$). Applying Theorem 4.3.1 and Corollary 4.3.1, we find that

$$\varphi^2(z) = W_2(z) - W_2(0) + \Phi_\varphi^{-1}\left(C_{\Omega_*}\left[\frac{\alpha_0 + w + G(w)}{\frac{1}{2}w}\right]\right)^\#(z)$$

As $0 \notin \Omega$, we find that $C_{\Omega_*}\left[\frac{\alpha_0+G(w)}{\frac{1}{2}w}\right] = 0$. Hence,

$$\begin{aligned} \varphi^2(z) &= W_2(z) - W_2(0) + 2\Phi_\varphi^{-1}(1) \\ &= c^2z^2 + 2cf_0z + 2. \end{aligned}$$

To solve for the undetermined coefficient f_0, we apply Equation 4.7,

$$\alpha_0 + w = h(w) = C_{\Omega_*}\left[\frac{1}{2}w\left(\frac{c^2}{\psi^2(w)} + \frac{2c\overline{f_0}}{\psi(w)} + 2\right)\right] = C_{\Omega_*}\left[w + c^2\overline{f_0} + O(w^{-1})\right] = w + c^2\overline{f_0}.$$

Hence, $f_0 = c^{-2}\overline{\alpha_0}$, and we find

$$\varphi^2(z) = c^2z^2 + 2c^{-1}\overline{\alpha_0}z + 2.$$

In particular, if $\Omega \in \mathrm{QD}_2(\alpha_0 + w)$ is a simply connected domain not containing 0 and φ is the associated Riemann map (uniquely associated via the normalization $\varphi(\infty) = \infty$, $\varphi'(\infty) = c > 0$), then φ satisfies the above equation in $\mathbb{D}_*$.

To determine uniqueness, suppose there were two such domains Ω_1 and Ω_2 with equal conformal radius $c > 0$ and associated Riemann maps φ_1 and φ_2. The prior equation tells us that $\varphi_1^2(z) = \varphi_2^2(z)$ for all $z \in \mathbb{D}_*$, so $1 = \left(\frac{\varphi_1(z)}{\varphi_2(z)}\right)^2$, which implies $\frac{\varphi_1(z)}{\varphi_2(z)} = \pm 1$ for all $z \in \mathbb{D}_*$. As φ_1 and φ_2 are non-zero and analytic in $\mathbb{D}_*$, we conclude that $\varphi_2 = \pm\varphi_1$. In particular, $\Omega_2 = \pm\Omega_1$. If $\Omega_2 = -\Omega_1$ then, applying the quadrature identity to the Cauchy kernel, we find that for each $w \in (\Omega_2)_*$,

$$h(w) = \oint_{\partial\Omega_2} \frac{h(\xi)d\xi}{\xi - w} = \int_{\Omega_2} \frac{|\xi|^2 dA(\xi)}{\xi - w} = -\int_{\Omega_1} \frac{|\xi|^2 dA(\xi)}{\xi - (-w)} = -\oint_{\partial\Omega_1} \frac{h(\xi)d\xi}{\xi - (-w)} = -h(-w)$$

□

Inner/Outer Factorization Identities for PQDs

As the title of this subsection suggests, this portion of the thesis is concerned with formulae obtained by decomposing the Riemann map into a product. In particular, the inner/outer factorization leveraged in our study of LQDs turns out to be equally fruitful for PQDs.

For the remainder of this discussion, fix $a > 0$ and let Ω be a simply connected domain with Riemann map φ admitting inner/outer factorization $\varphi = \varphi_{\text{in}}\varphi_{\text{out}}$ as in Theorem 2.4.2. In this case, we obtain the following simple characterization of simply connected PQDs:

Theorem 4.3.3. $\Omega \in QD_a$ *if and only if* φ_{out}^a *extends to a rational function.*

Proof of Theorem 4.3.3.

We will begin by considering the bounded case: Suppose $\Omega \in \mathrm{QD}_a(h)$ is a simply connected bounded domain with Riemann map $\varphi : \mathbb{D} \to \Omega$. By Theorem 2.4.2, $\varphi = \varphi_{\text{in}}\varphi_{\text{out}}$, where $\varphi_{\text{in}} = 1$ when $0 \notin \Omega$, $\varphi_{\text{in}} = b_{z_0}$ when $0 \in \Omega$ (z_0 the unique root of φ in $\mathbb{D}$), and $\varphi_{\text{out}} = (r^{\#})^{\frac{1}{a}}$, where

$$r(z) = \Phi_\varphi^{-1}\left(C_{\Omega_*}\left[|w|^{2a}\left(\frac{\varphi_{\text{in}} \circ \psi(w)}{w}\right)^a\right]\right)(z) + C,$$

for some $C \in \mathbb{C}$. By the coincidence equation for PQDs (e.g. Theorem 4.2.1), there exists $G \in \mathcal{A}(\Omega)$ such that $|w|^{2a} \doteq aw(h(w) + G(w))$. Substituting this into our formula for r, we obtain

$$r(z) = a\Phi_\varphi^{-1}\left(C_{\Omega_*}\left[(wh(w) + wG(w))\left(\frac{\varphi_{\text{in}} \circ \psi(w)}{w}\right)^a\right]\right)(z) + C,$$
$$= a\Phi_\varphi^{-1}\left(C_{\Omega_*}\left[wh(w)\left(\frac{\varphi_{\text{in}} \circ \psi(w)}{w}\right)^a\right]\right)(z) + C,$$

$(wG(w)\left(\frac{\varphi_{\text{in}}\circ\psi(w)}{w}\right)^a \in \mathcal{A}(\Omega)$, so it drops out). Moreover, note that $wh(w)\left(\frac{\varphi_{\text{in}}\circ\psi(w)}{w}\right)^a \in \mathcal{M}(\Omega)$, so its Cauchy projection onto $\mathcal{A}_0(\Omega_*)$ is rational. We conclude that r is rational, as the Faber transform of a rational function.

For the reverse direction, suppose $\varphi_{\text{out}} = (r^\#)^{\frac{1}{a}}$ with f rational. Then, as $|\varphi_{\text{in}}| \doteq 1$, we find that

$$\frac{1}{a}\overline{w}|w|^{2(a-1)} \doteq \frac{1}{aw}|\varphi|^{2a} \circ \psi(w) \doteq \frac{(rr^\#) \circ \psi(w)}{aw}$$

Hence,

$$S_a(w) := \frac{(rr^\#) \circ \psi(w)}{aw}$$

is meromorphic in Ω, extends continuously to $\partial\Omega$, and satisfies

$$S_a(w) \doteq \frac{1}{a}\overline{w}|w|^{2(a-1)}.$$

Thus S_a is a generalized Schwarz function for Ω, and Theorem 4.2.1 yields $\Omega \in \text{QD}_a$.

We will now consider the **unbounded** case: Suppose $\Omega \in \text{QD}_a(h)$ is a simply connected unbounded domain with Riemann map $\varphi : \mathbb{D}_* \to \Omega$. By Theorem 2.4.2, $\varphi = \varphi_{\text{in}}\varphi_{\text{out}}$, where $\varphi_{\text{in}}(z) = z$ when $0 \notin \Omega$, $\varphi_{\text{in}}(z) = zb_{z_0}(z)$ when $0 \in \Omega$ (z_0 the unique root of φ in $\mathbb{D}$), and $\varphi_{\text{out}} = (r^\#)^{\frac{1}{a}}$, where

$$r(z) = \Phi_\varphi^{-1}\left(C_{\Omega_*}\left[|w|^{2a}\left(\frac{\varphi_{\text{in}} \circ \psi(w)}{w}\right)^a\right]\right)(z) + C,$$

for some $C \in \mathbb{C}$. By the coincidence equation for PQDs (e.g. Theorem 4.2.1), there exists $G \in \mathcal{A}_0(\Omega)$ such that $|w|^{2a} \doteq aw(h(w) + G(w))$. Substituting this into our formula for r, we obtain

$$r(z) = a\Phi_\varphi^{-1}\left(C_{\Omega_*}\left[(wh(w) + wG(w))\left(\frac{\varphi_{\text{in}} \circ \psi(w)}{w}\right)^a\right]\right)(z) + C,$$
$$= a\Phi_\varphi^{-1}\left(C_{\Omega_*}\left[wh(w)\left(\frac{\varphi_{\text{in}} \circ \psi(w)}{w}\right)^a\right]\right)(z) + C,$$

$(wG(w)\left(\frac{\varphi_{\text{in}}\circ\psi(w)}{w}\right)^a$ is bounded and analytic in Ω, so it may be absorbed into C). Moreover, note that $wh(w)\left(\frac{\varphi_{\text{in}}\circ\psi(w)}{w}\right)^a \in \mathcal{M}(\Omega)$, so its Cauchy projection onto $\mathcal{A}(\Omega_*)$ is rational. We conclude that r is rational, as the Faber transform of a rational function. The argument for the reverse direction is entirely analogous to the argument in the bounded case. □

It follows immediately that if $\Omega \in \mathrm{QD}_a$ is simply connected, then there exists a rational function r such that

$$\varphi(z) = \varphi_{\text{in}}(z) r^{\#}(z)^{\frac{1}{a}}. \tag{4.8}$$

In particular, when Ω is bounded, $r \in \mathcal{A}(\mathbb{D}_*)$ and

$$\begin{aligned} \varphi_{\text{in}}(z) &= 1, && \text{when } 0 \notin \Omega, \\ \varphi_{\text{in}}(z) &= b_{z_0}(z), && \text{when } 0 \in \Omega. \end{aligned} \tag{4.9}$$

When Ω is unbounded, $r \in \mathcal{A}(\mathbb{D})$ and

$$\begin{aligned} \varphi_{\text{in}}(z) &= z, && \text{when } 0 \notin \Omega, \\ \varphi_{\text{in}}(z) &= z b_{z_0}(z), && \text{when } 0 \in \Omega, \end{aligned} \tag{4.10}$$

where z_0 is the unique root of φ when $0 \in \Omega$.

We furthermore obtain a formula for h in terms of this representation of φ, hence solving the direct problem for PQDs. In particular if $\Omega \in \mathrm{QD}_a(h)$ then by Theorem 4.2.1,

$$h(w) \doteq \frac{|w|^{2a}}{aw} - G(w) = \frac{|\varphi|^{2a} \circ \psi(w)}{aw} - G(w) \doteq \frac{\left(r r^{\#}\right) \circ \psi(w)}{aw} - G(w).$$

Noting that $C_{\Omega_*}[G] = 0$ and $C_{\Omega_*}[h] = h$, we find

$$h(w) = \frac{1}{a} C_{\Omega_*}\left[\frac{\left(r r^{\#}\right) \circ \psi(w)}{w}\right] \tag{4.11}$$

More generally, in the course of the proof of Theorem 4.3.3, we obtained an implicit solution to the inverse problem for PQDs. In the discussion to follow we sharpen this result, first for bounded PQDs, then for unbounded PQDs. By similar reasoning, we also obtain a solution to the direct problem for PQDs.

Bounded PQDs

The following theorem enables reconstruction of a simply connected bounded PQD from its quadrature function:

Theorem 4.3.4. *If $\Omega \in QD_a(h)$ is a bounded simply connected domain with Riemann map φ as in Equation 4.8 with $\varphi(0) = w_0$ then*

$$r = a \Phi_{\varphi}^{-1}\left(C_{\Omega_*}\left[w h(w) \left(\frac{\varphi_{\text{in}} \circ \psi(w)}{w}\right)^{a}\right]\right) + C, \tag{4.12}$$

where $C = \overline{w_0}^{a}$ when $0 \notin \Omega$ and $C = \frac{\overline{w_0}^{a}}{|z_0|^{a}}$ when $0 \in \Omega$.

Proof of Theorem 4.3.4. The proof of Theorem 4.3.3 already tells us that $\varphi(z) = \varphi_{\text{in}}(z) r^{\#}(z)^{\frac{1}{a}}$, where φ_{in} is given by Equation 4.9, and r is given by Equation 4.12 for some $C \in \mathbb{C}$. All that is left is to determine the value of C. When $0 \notin \Omega$, $\varphi_{\text{in}}(z) = 1$, so

$$w_0^a = \varphi^a(0) = r^{\#}(0) = \overline{r(\infty)}.$$

As $\Phi_\varphi^{-1} : \mathcal{A}_0(\Omega_*) \to \mathcal{A}_0(\mathbb{D}_*)$, we find that the Faber transform term is zero at ∞, so $r(\infty) = C$, and we find that $C = \overline{w_0}^a$. The argument when $0 \in \Omega$ is completely analogous. □

We can obtain by similar analysis a solution to the direct problem.

Theorem 4.3.5. *If $\Omega \in QD_a(h)$ is a bounded simply connected domain with Riemann map φ as in Equation 4.8 then*

$$h(w) = \frac{1}{aw} \Phi_\varphi \left(C_{\mathbb{D}_*} \left[r r^{\#} \right] \right)(w) + \frac{t}{w}, \tag{4.13}$$

where $t = \int_\Omega |w|^{2(a-1)} dA(w)$.

Proof of Theorem 4.3.5. Recall from the proof of Theorem 4.3.3 that

$$\frac{(rr^{\#}) \circ \psi(w)}{aw} := S_a(w)$$

is a generalized Schwarz function for Ω. Thus, by Theorem 4.2.1, $h(w) + G(w) \doteq \frac{(rr^{\#})\circ\psi(w)}{aw}$, where h is the quadrature function and $G \in \mathcal{A}(\Omega)$. Hence, by the definition of the Faber transform,

$$C_{\Omega_*}[wh(w)] = \frac{1}{a} C_{\Omega_*}\left[(rr^{\#}) \circ \psi(w)\right] - C_{\Omega_*}[wG(w)] = \Phi_\varphi \left(C_{\mathbb{D}_*} \left[rr^{\#} \right] \right)(w).$$

Moreover, note that if $w \in \Omega_*$, then

$$C_{\Omega_*}[wh(w)] = \oint_{\partial\Omega_*} \frac{\xi h(\xi)}{\xi - w} d\xi = -\oint_{\partial\Omega} h(\xi) d\xi + w \oint_{\partial\Omega_*} \frac{h(\xi)}{\xi - w} d\xi$$

Applying the quadrature identity to the first term, and Cauchy's integral formula to the second, we obtain

$$C_{\Omega_*}[wh(w)] = -\int_\Omega |\xi|^{2(a-1)} dA(\xi) + wh(w)$$

Thus,

$$wh(w) = C_{\Omega_*}[wh(w)] + \int_\Omega |\xi|^{2(a-1)} dA(\xi) = \Phi_\varphi \left(C_{\mathbb{D}_*} \left[rr^{\#} \right] \right)(w) + t.$$

□

Applying the fact that the inverse Faber transform takes poles to their image under $\psi = \varphi^{-1}$, we obtain the following corollary:

Corollary 4.3.2. Let h be a rational function with poles $\{p_k\}$ of respective orders $\{n_k\}$. Then the r in Theorem 4.3.4 is given by

$$r(z) = C + z^{-1}p^{\#}(z) + \sum_k \sum_{j=1}^{n_k} \frac{\beta_{k,j}}{(z-z_k)^j}. \tag{4.14}$$

for some constants $\{\beta_{k,j}\}$, $\varphi(z_k) = p_k$ and the same value of C as in the Theorem. If $\varphi(0) = p_j$ for some j, then the sum over k skips j and p is a polynomial of degree:

$$\deg(p) = \begin{cases} n_j - 1 & p_j \neq 0 \\ n_j - 2 & p_j = 0 \end{cases}$$

$p = 0$ otherwise.

Unbounded PQDs

We now state the analog of Theorem 4.3.4 for unbounded domains:

Theorem 4.3.6. *If $\Omega \in QD_a(h)$ is an unbounded simply connected domain then it has a Riemann map φ as in Equation 4.8 with*

$$r = a\Phi_\varphi^{-1}\left(C_{\Omega_*}\left[wh(w)\left(\frac{\varphi_{\text{in}} \circ \psi(w)}{w}\right)^a\right]\right) + C, \tag{4.15}$$

where, $C = \frac{a}{c^a}t$ when $0 \notin \Omega$ and $C = \frac{a}{|cz_0|^a}t$ when $0 \in \Omega$, and $t = \int_{\Omega^c} |w|^{2(a-1)}dA(w)$.

Proof of Theorem 4.3.6. The proof of Theorem 4.3.3 already tells us that Equation 4.15 holds for some $C \in \mathbb{C}$. We first consider the case in which $0 \in \Omega$, so that $\varphi(z) = zb_{z_0}(z)r^{\#}(z)^{\frac{1}{a}}$. Recall that $c > 0$ is the conformal radius of Ω, so

$$c^a|z_0|^a = \lim_{|z|\to\infty}\left(\frac{\varphi(z)}{zb_{z_0}(z)}\right)^a = r^{\#}(\infty) = \overline{r(0)}.$$

Hence,

$$C = |cz_0|^a - a\Phi_\varphi^{-1}\left(C_{\Omega_*}\left[wh(w)\left(\frac{\varphi_{\text{in}} \circ \psi(w)}{w}\right)^a\right]\right)(0).$$

Applying Equation 2.8 and changing variables, we obtain

$$a\Phi_\varphi^{-1}\left(C_{\Omega_*}\left[wh(w)\left(\frac{\varphi_{\text{in}} \circ \psi(w)}{w}\right)^a\right]\right)(0) = \oint_{\partial\Omega_*} a\xi h(\xi)\left(\frac{\varphi_{\text{in}} \circ \psi(\xi)}{\xi}\right)^a \frac{\psi'(\xi)}{\psi(\xi)}d\xi.$$

Substituting the generalized coincidence equation and changing variables back yields

$$\begin{aligned} a\Phi_\varphi^{-1}\left(C_{\Omega_*}\left[wh(w)\left(\frac{\varphi_{\text{in}}\circ\psi(w)}{w}\right)^a\right]\right)(0) &= \oint_{\partial\Omega_*}(|\xi|^{2a}-a\xi G(\xi))\left(\frac{\varphi_{\text{in}}\circ\psi(\xi)}{\xi}\right)^a\frac{\psi'(\xi)}{\psi(\xi)}d\xi \\ &= \oint_{\partial\mathbb{D}}(|\varphi_{\text{out}}(\xi)|^{2a}-a\varphi(\xi)G\circ\varphi(\xi))\left(\frac{1}{\varphi_{\text{out}}(\xi)}\right)^a\frac{1}{\xi}d\xi \\ &= \oint_{\partial\mathbb{D}}\frac{\varphi_{\text{out}}^{\#}(\xi)^a}{\xi}d\xi - a\oint_{\partial\mathbb{D}}\frac{\varphi(\xi)G\circ\varphi(\xi)}{\xi\varphi_{\text{out}}(\xi)^a}d\xi \end{aligned}$$

Note that $\varphi_{\text{out}} \in \mathcal{A}(\mathbb{D}_*)$ is bounded and non-zero in $\mathbb{D}_*$, so $\varphi_{\text{out}}^{\#}(z)^a \in \mathcal{A}(\mathbb{D})$. Hence, the first integral is precisely $\varphi_{\text{out}}^{\#}(0)^a = r(0) = |cz_0|^a$. For the second integral, we note that the integrand is likewise analytic in $\mathbb{D}_*$ except possibly at ∞. Expanding G about ∞, we find that $G(w) = tw^{-1} + O(w^{-2})$, where $t = \int_{\Omega^c}|w|^{2(a-1)}dA(w)$. Hence,

$$\begin{aligned} C &= a\oint_{\partial\mathbb{D}}\frac{\varphi(\xi)G\circ\varphi(\xi)}{\xi\varphi_{\text{out}}(\xi)^a}d\xi \\ &= -a\operatorname*{Res}_{\xi=\infty}\frac{t+O(\xi^{-1})}{\xi\varphi_{\text{out}}(\xi)^a} \\ &= a\lim_{|\xi|\to\infty}\frac{t+O(\xi^{-1})}{\varphi_{\text{out}}(\xi)^a} \\ &= \frac{a}{|cz_0|^a}t \end{aligned}$$

The proof in the non-singular case ($0 \notin \Omega$) is completely analogous. □

We can obtain by similar analysis a solution to the direct problem.

Theorem 4.3.7. *If $\Omega \in QD_a(h)$ with Riemann map φ as in Equation 4.8 then*

$$h(w) = \frac{1}{aw}\Phi_\varphi\left(C_{\mathbb{D}}\left[rr^{\#}\right]\right)(w) - \frac{t}{w}, \tag{4.16}$$

where $t = \int_{\Omega^c}|w|^{2(a-1)}dA(w)$.

Proof of Theorem 4.3.7. Recall from the proof of Theorem 4.3.3 that $\frac{(rr^{\#})\circ\psi(w)}{aw} := S_a(w)$ is a generalized Schwarz function for Ω. Thus, by Theorem 4.2.1, $h(w)+G(w)\doteq\frac{(rr^{\#})\circ\psi(w)}{aw}$, where h is the quadrature function and $G \in \mathcal{A}_0(\Omega)$. Hence, by the definition of the Faber transform,

$$wh(w) = C_{\Omega_*}[wh(w)] = \frac{1}{a}C_{\Omega_*}\left[(rr^{\#})\circ\psi(w)\right] - C_{\Omega_*}[wG(w)] = \Phi_\varphi\left(C_{\mathbb{D}_*}\left[rr^{\#}\right]\right)(w) - t.$$

Where the last equality follows from expansion of Equation 4.5 about ∞, which demonstrates that $wG(w) = t + g(w)$, where $t = \int_{\Omega^c}|w|^{2(a-1)}dA(w)$ and $g \in \mathcal{A}_0(\Omega)$, so

$$C_{\Omega_*}[wG(w)] = C_{\Omega_*}[t+g(w)] = t.$$

□

Example 4.3.1. Consider the map $\varphi : \mathbb{D}_* \to \widehat{\mathbb{C}}$ given by $\varphi(z) = cz\left(1 - \frac{\gamma}{z}\right)^{\frac{1}{a}}$, where $|\gamma| < 1$ and $a, c > 0$. Note that $\Omega =: \varphi(\mathbb{D}_*)$ is an unbounded domain not containing 0 on account of the assumption that $|\gamma| < 1$. Hence, when φ is univalent, it is a Riemann map for Ω with $\varphi^a_{\text{out}}(z) = r^\#(z) = c^a\left(1 - \frac{\gamma}{z}\right)$ rational so Theorem 4.3.3 tells us that $\Omega \in \mathrm{QD}_a$. We may now apply Theorem 4.3.7 to solve the direct problem of finding the quadrature function associated to Ω:

$$
\begin{aligned}
h(w) &= \frac{1}{aw}\Phi_\varphi\left(C_{\mathbb{D}}\left[c^a\left(1 - \frac{\gamma}{z}\right)c^a\left(1 - \overline{\gamma}z\right)\right]\right)(w) - \frac{t}{w} \\
&= \frac{c^{2a}}{aw}\Phi_\varphi\left(C_{\mathbb{D}}\left[1 + |\gamma|^2 - \frac{\gamma}{z} - \overline{\gamma}z\right]\right)(w) - \frac{t}{w} \\
&= \frac{c^{2a}}{aw}\Phi_\varphi\left(1 + |\gamma|^2 - \overline{\gamma}z\right)(w) - \frac{t}{w} \\
&= \frac{c^{2a}}{aw}\left(1 + |\gamma|^2 - \overline{\gamma}(wc^{-1} + f_0)\right) - \frac{t}{w} \\
&= -\frac{c^{2a-1}}{a}\overline{\gamma} + \frac{c_1}{w}.
\end{aligned}
$$

However, $h \in \mathcal{A}(\Omega_*)$ and $0 \in \Omega_*$, so $c_1 = 0$ and we find that $\Omega \in \mathrm{QD}_a(\alpha)$, for $\gamma = -\frac{\overline{\alpha}a}{c^{2a-1}}$. We will characterize the univalence of φ in §4.5. Several examples of Ω are provided in Figure 4.2. Note that in the limit as $|\gamma| \to 1$, $\partial\Omega$ approaches 0, forming a "corner" there.

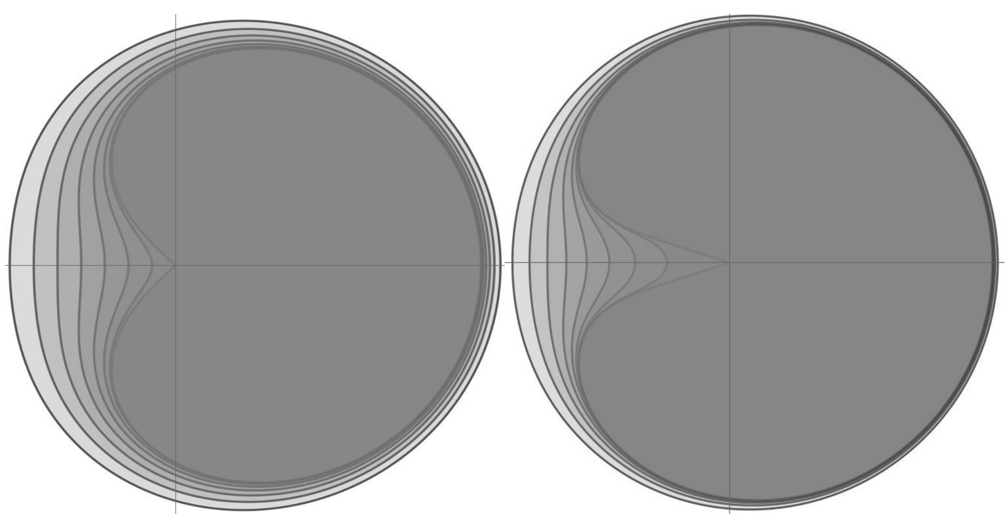

Figure 4.2: Two families of simply connected unbounded $\Omega \in \mathrm{QD}_a\left(\frac{1}{2}\right)$ (the complements of the shaded regions) with $a = 2, 5$ (Example 4.3.1)

We furthermore obtain a detailed characterization of the pole structure of r in terms of h.

Corollary 4.3.3. Let h be a rational function with poles $\{p_k\}$ of respective orders $\{n_k\}$, and a pole at ∞ of order n. Then the r in Theorem 4.3.6 is given by

$$r(z) = C + zp(z) + z \sum_k \sum_{j=1}^{n_k} \frac{\beta_{k,j}}{(z - z_k)^j}, \tag{4.17}$$

for some constants $\{\beta_{k,j}\}$, $\varphi(z_k) = p_k$, a polynomial p of degree n ($p = 0$ if $h(\infty) = 0$); and C as in the theorem. Moreover, if h has a pole at zero then the corresponding term in the above sum is of one degree lower. In particular, *the poles of h are the images of those of r under φ.*

4.4 Characterization of One-Point PQDs

In the prior discussion, we showed that simply connected PQDs admit representations in terms of rational functions. In the following analysis, we characterize this representation for one-point PQDs of arbitrary order. Propositions 4.6.1 and 4.6.2 cover the case in which the quadrature node is finite (i.e. the quadrature function is a rational function with a unique pole). Proposition 4.6.3 covers the case in which the quadrature node is at ∞ (i.e. the quadrature function is a polynomial).

We begin with a classification of simply connected one-point PQDs of order one with coefficient $\alpha > 0$, $\mathrm{QD}_a\left(\frac{\alpha}{w-w_0}\right)$. There are several cases to consider:

1. Ω is bounded:
   a) $0 \notin \Omega$,
   b) $0 \in \Omega$ and $w_0 \neq 0$,
   c) $0 \in \Omega$ and $w_0 = 0$,
2. Ω is unbounded:
   a) $0 \notin \Omega$,
   b) $0 \in \Omega$, but $w_0 \neq 0$.[1]

Case 1 is covered by [9] for integer a and summarized by the following theorem:

Theorem 4.4.1. *For each $n \in \mathbb{Z}_+$, $w_0 \in \mathbb{C}$, and $\alpha > 0$, there exists a bounded simply connected $\Omega \in QD_n\left(\frac{\alpha}{w-w_0}\right)$. In addition,*

(a) If $|w_0|^{2n} \geq n^2\alpha$, then Ω is an nth root of $\mathbb{D}_{n\sqrt{\alpha}}(w_0^n)$, specifically the connected component containing w_0.

(b) If $w_0 = 0$, then $\Omega = \mathbb{D}_{\sqrt[2n]{\alpha n}}(0)$,

[1] By symmetry, Ω cannot be unbounded if $w_0 = 0$.

(c) If $0 < |w_0|^{2n} < n^2\alpha$*, then* Ω *is the image of the unit disk under a map of the form* $\varphi(z) = \dfrac{\delta z}{(1-z\overline{z_0})^{\frac{1}{n}}}$*, where* δ *is a constant and* $\varphi(z_0) = w_0$*.*

Cases (a) *and* (c) *correspond to* $0 \notin \Omega$ *and* $0 \in \Omega$ *respectively.*

By applying the following univalence criterion (Lemma 4.4.1), we show furthermore that in case (c), φ is univalent in $\mathbb{D}$ if and only if $|z_0| < 1$. An easy corollary of Theorem 6.6 in [26] is that

Lemma 4.4.1. An analytic function $\varphi : \mathbb{D} \to \widehat{\mathbb{C}}$ for which $\varphi(0) = 0$ is univalent and starlike with respect to the origin if and only if

$$\mathrm{Re}\left(z\frac{\varphi'(z)}{\varphi(z)}\right) > 0$$

for all $z \in \mathbb{D}$.

In our case, $\varphi(z) = \frac{\delta z}{(1-z\overline{z_0})^{\frac{1}{n}}}$ is analytic in $\mathbb{D}$ if and only if $|z_0| < 1$ so, wlog replacing $\overline{z_0}$ with $|z_0|$ and applying the lemma and partial fractions we obtain

$$n\mathrm{Re}\left(z\frac{\varphi'(z)}{\varphi(z)}\right) = \mathrm{Re}\left(z\frac{(\varphi^n)'(z)}{\varphi^n(z)}\right) = n - 1 + \mathrm{Re}\left(\frac{1}{1-z|z_0|}\right).$$

Taking $z|z_0| = re^{i\theta}$ and noting that $n \geq 1$, we find

$$n\mathrm{Re}\left(z\frac{\varphi'(z)}{\varphi(z)}\right) \geq \frac{1}{2}\left(1 + \frac{1-r^2}{r^2 - 2r\cos(\theta) + 1}\right) \geq \frac{1}{2}\left(1 + \frac{1-r^2}{r^2+2r+1}\right) > 0$$

for all $0 \leq r < 1$. The desired conclusion follows.

We now consider the unbounded case. As in the discussion of classical one-point QDs (§3.3), it is instructive to consider the situation from the perspective of potential theory. In particular, Propositions 4.4.1 and 4.4.2 to follow characterize the relationship between these two perspectives.

Proposition 4.4.1. If K is a local droplet of a potential $Q(w) = \frac{|w|^{2a}}{a^2} - 2\mathrm{Re}(H(w))$ for some $a > 0$ and H with rational derivative, then K^c is a disjoint union of PQDs for which $h := H'$ is the sum of their quadrature functions.

This is an easy consequence of the existence of the coincidence equation for local droplets (Equation 2.1) and Theorem 4.2.1.

Proposition 4.4.2. If $K \subset \mathbb{C}$ is a compact subset such that K^c is a finite disjoint union of PQDs, $\Omega_j \in \mathrm{QD}_a(h_j)$, $a > 0$, then K is a local droplet. Furthermore, the external potential associated to the droplet is given by

$$Q(w) = \frac{|w|^{2a}}{a^2} - 2\mathrm{Re}(H(w)),\ H(w) = 2\mathrm{Re}\left(\int_{w_l}^{w} h(\xi)d\xi\right) + \mathrm{const}(w_l) \text{ in some open nbhd of } K_l$$

(where $h = \sum h_j$ and w_l is any point in the lth connected component of K, K_l). In particular, Q is unique up to several additive constants.

Proof of Proposition 4.4.2. Let $K^c = \Omega_\infty \cup \bigcup_{j=1}^N \Omega_j$ be a disjoint union of PQDs with respective quadrature functions h_j. Also let $h = \sum_j h_j$. Applying Frostman's theorem (Proposition 2.1.2), it is sufficient to construct an open nbhd $O \supset K$ and harmonic $H : O \to \mathbb{R}$ such that $\frac{\partial H}{\partial w} = h$ and

$$\frac{|w|^{2a}}{a^2} - H(w) + U^\mu(w) = 0, \ \forall w \in K, \tag{4.18}$$

where $d\mu = \mathbb{1}_K \frac{\Delta Q}{2} dA$. We shall choose O as an $\epsilon-$nbhd of K for a to be determined value of ϵ. We require that O satisfies each of the following

1. h is holomorphic in O,
2. each connected component of O contains precisely one connected component of K,
3. every loop in O is homotopic to a loop in K.

By assumption, ∂K has finitely many singular points. Thus there exists $\epsilon > 0$ s.t. (2) and (3) hold. Note that, for each Ω_j and each $w \in (\Omega_j)_*$, the quadrature identity tells us that

$$C_Q^{\Omega_j}(w) = \int_{\Omega_j} \frac{|\xi|^{2(a-1)}}{w-\xi} dA(\xi) = \oint_{\partial\Omega_j} \frac{h_j(\xi)}{w-\xi} d\xi = \oint_{\partial(\Omega_j)_*} \frac{h_j(\xi)}{\xi - w} d\xi = h_j(w),$$

and this extends to $\partial\Omega_j$. Thus, for each $w \in \bigcap_j \Omega_j^c = K$,

$$C_Q^{\Omega}(w) = \int_{\Omega} \frac{|\xi|^{2(a-1)}}{w-\xi} dA(\xi) = \sum_j \int_{\Omega_j} \frac{|\xi|^{2(a-1)}}{w-\xi} dA(\xi) = \sum_j C_Q^{\Omega_j}(w) = \sum_j h_j(w) = h(w).$$

But then note that

$$\begin{aligned} C_Q^{\mathbb{C}}(w) = \int_{\mathbb{C}} \frac{|\xi|^{2(a-1)}}{w-\xi} dA(\xi) = \lim_{R\to\infty} \int_{\mathbb{D}_R} \frac{|\xi|^{2(a-1)}}{w-\xi} dA(\xi) &= \frac{1}{a}\overline{w}|w|^{2(a-1)} - \frac{1}{a} \lim_{R\to\infty} \oint_{\partial\mathbb{D}_R} \frac{\overline{\xi}|\xi|^{2(a-1)}}{\xi - w} d\xi \\ &= \frac{1}{a}\overline{w}|w|^{2(a-1)} - \frac{1}{a} \lim_{R\to\infty} R^{2a} \oint_{\partial\mathbb{D}_R} \frac{\xi^{-1}}{\xi - w} d\xi = \frac{1}{a}\overline{w}|w|^{2(a-1)}. \end{aligned}$$

So $C_Q^K(w) + C_Q^\Omega(w) = C_Q^{\mathbb{C}}(w) = \frac{1}{a}\overline{w}|w|^{2(a-1)}$. Thus $h(w) = \frac{1}{a}\overline{w}|w|^{2(a-1)} - C_Q^K(w)$ on K. Then note that if γ is a loop in K,

$$\begin{aligned} 2\mathrm{Re}\left(\oint_\gamma h(w)dw\right) &= \oint_\gamma \left(\frac{1}{a}\overline{w}|w|^{2(a-1)} - C_Q^K(w)\right) dw + \oint_\gamma \left(\overline{\frac{1}{a}\overline{w}|w|^{2(a-1)}} - \overline{C_Q^K(w)}\right) d\overline{w} \\ &= \oint_\gamma \partial \left(\frac{|w|^{2a}}{a^2} + U^\mu(w)\right) + \oint_\gamma \overline{\partial} \left(\frac{|w|^{2a}}{a^2} + U^\mu(w)\right) \\ &= \oint_\gamma d \left(\frac{|w|^{2a}}{a^2} + U^\mu(w)\right) = 0. \end{aligned}$$

So, by Morera's theorem, h is holomorphic in K and thus in $\mathcal{O}$ because h is a sum of quadrature functions on the components of K^c, so, by (3), this result extends to $\mathcal{O}$.

As a consequence, if we pick a point w_l in each connected component $\mathcal{O}_l$ of $\mathcal{O}$ and set

$$H(w) := 2\mathrm{Re}\left(\int_{w_l}^{w} h(\xi)d\xi\right) + \frac{|w_l|^{2a}}{a^2} + U^{\mu}(w_l),\ \forall w \in \mathcal{O}_l,$$

is a real, well-defined harmonic function in $\mathcal{O}$ and $\frac{\partial H}{\partial w} = h$ on K. Finally, we need to verify Equation 4.18. By construction, the identity holds at each w_l. Furthermore, we have

$$\frac{\partial}{\partial w}\left(\frac{|w|^{2a}}{a^2} - H(w) + U^{\mu}(w)\right) = \frac{1}{a}\overline{w}|w|^{2(a-1)} - h(w) - C^K(w) = 0 \ \text{ on } K,$$

and

$$\frac{\partial}{\partial \overline{w}}\left(\frac{|w|^{2a}}{a^2} - H(w) + U^{\mu}(w)\right) = 0 \ \text{ on } K.$$

Equation 4.18 follows.

An additional consequence of this (from Frostman's theorem 2.1.2 and Equation 4.18) is that K is a local droplet of the potential $Q(w) = \frac{|w|^{2a}}{a^2} - H(w)$. □

Simple One-Point PQDs

By Proposition 4.4.2, if $\Omega \in \mathrm{QD}_a\left(\frac{\alpha}{w-w_0}\right)$ is unbounded and simply connected, then Ω^c is a local droplet of the potential

$$Q(w) = \frac{|w|^{2a}}{a^2} - 2\mathrm{Re}(\alpha \ln(w - w_0)).$$

Thus one might expect a $1-$parameter family of PQDs $\{\Omega_t\}_t$, parameterized by the weighted area of the droplets in the corresponding weighted Hele-Shaw chain. This turns out to be the case. However, the situation is more complicated than the classical case discussed in §3.3 on account of the singularity of the weight at 0 - this phenomenon also manifests itself in Propositions 4.6.3, 4.6.1, and 4.6.2. In particular, there are apparently four distinct families to consider, depicted in Figure 4.3.

If $\alpha > 0$ then there seems to be a unique simply connected $1-$parameter family

$$\{\Omega_t\}_{0<t\le t_*} \overset{?}{=} \left\{\Omega \in \mathrm{QD}_a\left(\frac{\alpha}{w - w_0}\right) : \Omega \text{ is unbounded}\right\},$$

where $t_* = \frac{w_0^a}{a}(w_0^a + 2a\sqrt{\alpha}) + (a-1)\alpha$.

The family and the two phases can be characterized in terms of their Riemann maps, given by Equations 4.19 and 4.20 respectively.

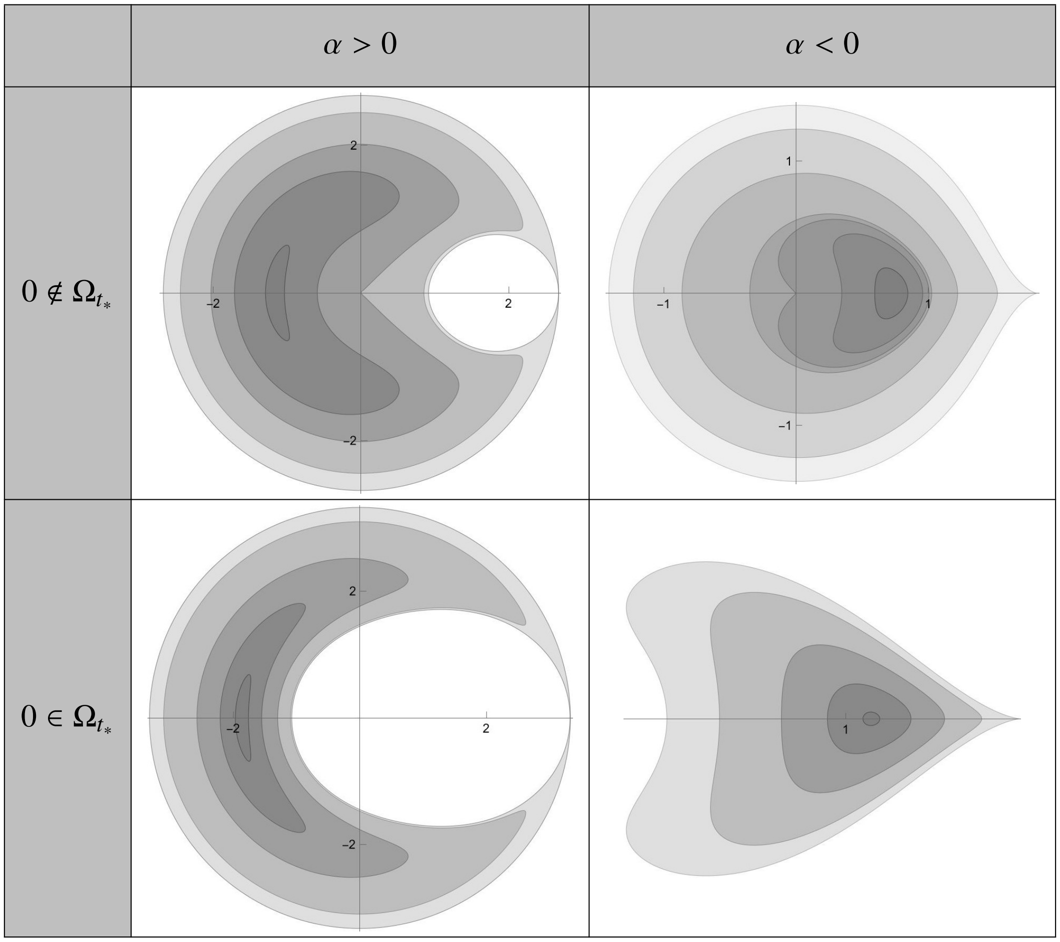

Figure 4.3: The families of one-point unbounded PQDs (complements of the shaded regions) in $\mathrm{QD}_2\left(\frac{\alpha}{w-2}\right)$ with $\alpha = \frac{5}{2}$, $\alpha = \frac{25}{2}$ in column 1 and $\alpha = -\frac{1}{4}$, $\alpha = -\frac{3}{5}$ in column 2. Rows 1 and 2 were obtained using Theorems 4.4.2 and 4.4.3 respectively.

Theorem 4.4.2. *Fix $a > 0$ and let $0 \notin \Omega$ be an unbounded simply connected domain with conformal radius $c > 0$. Then there exist $\alpha, w_0 \in \mathbb{C} \setminus \{0\}$ for which $\Omega \in QD_a\left(\frac{\alpha}{w-w_0}\right)$ if and only if there exists $z_0 \in \mathbb{D}_*$ such that $\Omega = \varphi(\mathbb{D}_*)$, where $\varphi : \mathbb{D}_* \to \widehat{\mathbb{C}}$ is a univalent conformal map of the form*

$$\varphi(z) = cz\left(1 + \frac{|z_0|^2 - 1}{z\overline{z_0} - 1}\left(\beta^a - 1\right)\right)^{\frac{1}{a}}, \tag{4.19}$$

for which $\alpha = \frac{c^{2a}}{a^2}(1 - \overline{\beta}^a)\left(|z_0|^2(1 + \beta^a(a-1)) - a\beta^a\right)$ and $\left|1 - (|z_0|^2 - 1)(\beta^a - 1)\right| > |z_0|$, where $\beta = \frac{w_0}{cz_0}$. Also note that $\varphi(z_0) = w_0$.

Proof of Theorem 4.4.2.

The equivalence follows directly from Proposition 4.6.2. To obtain Equation 4.19, apply Proposition 4.6.2 so that $\varphi(z) = cz\left(\frac{z-z_1}{z-\overline{z_0}^{-1}}\right)^{\frac{1}{a}}$, with $\varphi(z_0) = w_0$, which implies $z_1 = z_0 - \frac{|z_0|^2-1}{\overline{z_0}}\left(\frac{w_0}{cz_0}\right)^a$. Substitution and simplification yields Equation 4.19. The final relation follows straightforwardly from Equation 4.16. □

Theorem 4.4.3. *Fix $1 \neq a > 0$, and let $0 \in \Omega$ be an unbounded simply connected domain with conformal radius $c > 0$. Then there exist $\alpha, w_0 \in \mathbb{C} \setminus \{0\}$ for which $\Omega \in QD_a\left(\frac{\alpha}{w-w_0}\right)$ if and only if there are $z_0, z_1 \in \mathbb{D}_*$ such that $\Omega = \varphi(\mathbb{D}_*)$, where $\varphi : \mathbb{D}_* \to \widehat{\mathbb{C}}$ is a conformal map of the form*

$$\varphi(z) = cz|z_1|b_{z_1}(z)\left(1 + \frac{|z_0|^2-1}{z\overline{z_0}-1}\left(\beta^a - 1\right)\right)^{\frac{1}{a}}, \tag{4.20}$$

for which $\beta = \frac{w_0}{cz_0}\frac{z_0-\overline{z_1}^{-1}}{z_0-z_1}$, and satisfies the relation

$$\alpha = \frac{c^{2a}}{a^2}\frac{|z_1|^{2(a-1)}}{|z_0|^2-1}\left(a\overline{z_0}(z_0(\overline{z_1}z_0 - 2) + z_1) - \frac{|z_0-z_1|^2}{1-|z_0|^{-2}}\right).$$

Also note that $\varphi(z_0) = w_0$, and $\varphi(z_1) = 0$.

Proof of Theorem 4.4.3.

The equivalence follows directly from Proposition 4.6.2. To obtain Equation 4.20 note that, by Proposition 4.6.2,

$$\varphi(z) = cz|z_1|b_{z_1}(z)\left(\frac{z-\overline{z_1}^{-1}}{z-\overline{z_0}^{-1}}\right)^{\frac{1}{a}},$$

with $\varphi(z_0) = w_0$, which implies $\overline{z_0}^{-1} = z_0 - \frac{z_0\overline{z_1}-1}{\overline{z_1}}\left(\frac{cz_0\overline{z_1}}{w_0}\frac{z_0-z_1}{z_0\overline{z_1}-1}\right)^a = z_0 - \frac{z_0\overline{z_1}-1}{\overline{z_1}}\beta^a$. Substitution and simplification yields Equation 4.20. The final relation follows from Equation 4.16. □

Before continuing on to our discussion of Monomial PQDs, we state the following univalence criterion particularly applicable to PQDs.

Proposition 4.4.3. Take $a > 0$, $k \in \mathbb{Z}_+$, $\{z_j\}_{j=1}^k$ nonzero constants, and assume $\varphi : \mathbb{D}_* \to \widehat{\mathbb{C}}$ is given by $\varphi(z) = z\left(\left(1 - \frac{z_1}{z}\right)\ldots\left(1 - \frac{z_k}{z}\right)\right)^{\frac{1}{a}}$. Then,

1. If $k \leq 2a$ then φ is univalent if and only if each $|z_j| \leq 1$. In this case, $\varphi(\mathbb{D}_*)$ is starlike with respect to infinity.

2. If $k > 2a$, then φ is univalent and $\varphi(\mathbb{D}_*)$ is starlike with respect to infinity if and only if $|z_j| \leq 1$ and

$$a + \mathrm{Re}\left(\sum_{j=1}^k \frac{z_j}{z - z_j}\right) > 0, \quad \forall z \in \mathbb{D}_*.$$

The proof of Proposition 4.4.3 is in Appendix A.3

4.5 Classification of Monomial PQDs

In this section, we attempt to classify the set of simply connected *monomial PQDs*: domains $\Omega \in \mathrm{QD}_a(\alpha k w^{k-1})$ for some $\alpha \in \mathbb{C}$ and $k \in \mathbb{Z}_+$. We begin with the $k = 1$ case: $\Omega \in \mathrm{QD}_a(\alpha)$, referred to as *basic monomial PQDs*. In this case, we obtain a complete classification, with Theorem 4.5.1 covering the non-singular case, and Theorem 4.5.2 covering the singular case. Using symmetry arguments and an appropriate change of variables, we then obtain a classification of rotationally symmetric monomial PQDs for $k > 1$ (Theorem 4.5.3). Finally, in Example 4.5.1, we provide an explicit example of non-uniqueness for PQDs: We exhibit a domain Ω such that $-\Omega, \Omega \in \mathrm{QD}_2(2\alpha w)$ and $-\Omega \cap \Omega \neq \emptyset$. Hence, demonstrating that the uniqueness conjecture for classical quadrature domains[2] does not generalize to PQDs.

Classification of Basic Monomial PQDs

In Example 4.3.1, we showed that the image Ω of $\mathbb{D}_*$ under the map

$$\varphi(z) = cz\left(1 - \frac{\gamma}{z}\right)^{\frac{1}{a}} \tag{4.21}$$

is a PQD with quadrature function $\alpha = -\frac{c^{2a-1}}{a}\overline{\gamma}$ whenever $|\gamma| < 1$, $a, c > 0$, and φ is univalent in $\mathbb{D}_*$.

Using this result as a starting point, our aim in this section is to fully classify the simply connected domains $\Omega \in \mathrm{QD}_a(\alpha)$. We refer to such domains as *basic monomial PQDs*. It turns out that all such domains not containing zero admit a Riemann map in the form of Equation 4.21. The situation is slightly more complicated when $0 \in \Omega$, so we deal with this case last.

Non-Singular Basic Monomial PQDs

Theorem 4.5.1 below provides a characterization of basic monomial PQDs not containing zero.

Theorem 4.5.1. *Take $a > 0$ and $\alpha \in \mathbb{C} \setminus \{0\}$. There exists a simply connected domain Ω, not containing zero, of conformal radius c for which $\Omega \in QD_a(\alpha)$ if and only if either*

1. $0 < a < \frac{1}{2}$ *and* $|\gamma| \leq \frac{a}{1-a}$,
2. $a > \frac{1}{2}$ *and* $|\gamma| \leq 1$,
3. $a = \frac{1}{2}$ *and* $|\alpha| \leq 2$.

[2] The conjecture states that simply connected quadrature domains are uniquely associated to their quadrature function (modulo conformal radius in the unbounded case).

where $\gamma = -\frac{a\overline{\alpha}}{c^{2a-1}}$. *In this case,* Ω *is unique modulo conformal radius, and* $\Omega = \varphi(\mathbb{D}_*)$ *with* φ *univalent in* $\mathbb{D}_*$ *and given by Equation 4.21.*

Proof of Theorem 4.5.1. We proved in Example 4.3.1 that if φ (Equation 4.21) is a Riemann map for Ω, then $\Omega \in \mathrm{QD}_a(\alpha)$, where $\alpha = -\frac{c^{2a-1}}{a}\overline{\gamma}$.

Conversely, suppose $\Omega \not\ni 0$ is a simply connected domain in $\mathrm{QD}_a(\alpha)$ for some $a > 0$ and $\alpha \in \mathbb{C}\setminus\{0\}$ (necessarily unbounded because ∞ is a quadrature node). Then by Theorem 4.3.6 there exists a rational function r for which $\varphi(z) = zr^{\#}(z)^{\frac{1}{a}}$ is a Riemann map for Ω and

$$r = a\Phi_{\varphi}^{-1}\left(C_{\Omega_*}\left[\alpha w\left(\frac{\psi(w)}{w}\right)^a\right]\right) + \frac{a}{c^a}t.$$

Also note that the argument of the Faber transform grows linearly about ∞, so there exists $c_0 \in \mathbb{C}$ s.t. $w\left(\frac{\psi(w)}{w}\right)^a - \frac{w+c_0}{c^a} \in \mathcal{A}_0(\Omega)$, which implies

$$C_{\Omega_*}\left[\alpha w\left(\frac{\psi(w)}{w}\right)^a\right] = \frac{\alpha}{c^a}C_{\Omega_*}\left[w + c_0\right] + \alpha C_{\Omega_*}\left[w\left(\frac{\psi(w)}{c^{-1}w}\right)^a - \frac{w + c_0}{c^a}\right] = \frac{\alpha}{c^a}(w + c_0).$$

Substituting this into the above expression and computing the Faber transform, we obtain

$$r^{\#}(z) = \frac{a\overline{\alpha}}{c^a}\Phi_{\varphi}^{-1}\left(w + c_0\right)^{\#}(z) + \frac{a}{c^a}t = a\overline{\alpha}\frac{cz^{-1} + \overline{f_0}}{c^a} + \frac{a}{c^a}t = \frac{a\overline{\alpha}}{c^{a-1}}z^{-1} + c_1.$$

Considering the asymptotics, $\varphi(z) = cz + O(1)$, we find that $c_1 = c^a$, so

$$\varphi(z) = cz\left(1 - \frac{\gamma}{z}\right)^{\frac{1}{a}}.$$

Now all there is to do is demonstrate that φ is univalent if and only if one of conditions 1-3 in the theorem is satisfied. By Proposition 4.4.3, when $a \geq \frac{1}{2}$, φ is univalent in $\mathbb{D}_*$ if and only if $|\gamma| \leq 1$. Furthermore, when $a = \frac{1}{2}$, γ is independent of c with $-2\gamma = \overline{\alpha}$, so φ is univalent if and only if $|\alpha| \leq 2$.

On the other hand, if $a < \frac{1}{2}$ then Proposition 4.4.3 tells us that φ is univalent in $\mathbb{D}_*$ if and only if $|\gamma| \leq 1$ and $a + \mathrm{Re}\left(\frac{\gamma}{z-\gamma}\right) > 0$ for all $z \in \mathbb{D}_*$. Rotating $\mathbb{D}_*$ by taking $z \mapsto -\frac{\gamma}{|\gamma|}z$ yields $a - |\gamma|\mathrm{Re}\left(\frac{1}{z+|\gamma|}\right)$ which is minimized as $z \to 1$. Thus we obtain the condition $|\gamma| \leq \frac{a}{1-a}$. Or, equivalently $0 < c \leq c_*$, where $c_* = \sqrt[2a-1]{|\alpha|(1-a)}$. Thus when $0 < a < \frac{1}{2}$, φ is univalent if and only if $|\gamma| \leq \frac{a}{1-a}$. □

Singular Basic Monomial PQDs

The additional complexity in this case is owed to the fact that when $0 \in \Omega$, the formula for the associated Riemann map picks up a Blaschke factor with parameter z_0, the unique root of φ in $\mathbb{D}_*$. φ nevertheless admits a relatively simple representation.

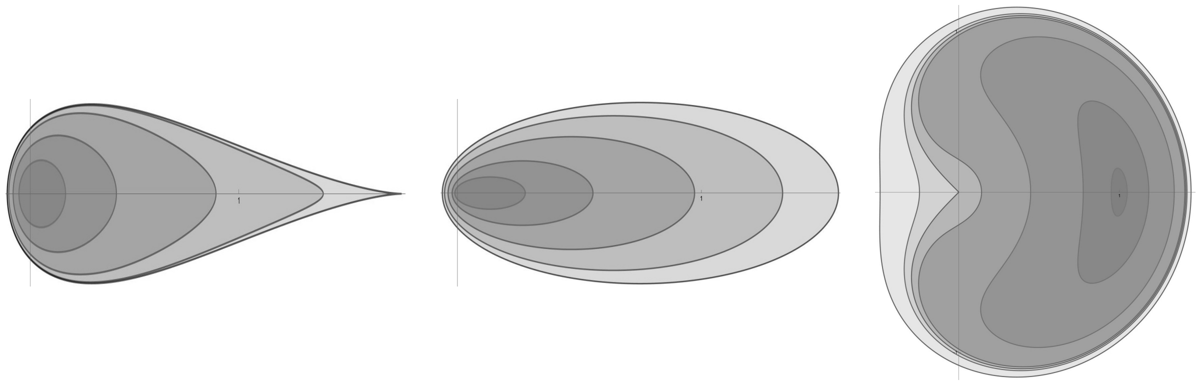

Figure 4.4: Example families of $\Omega \in \mathrm{QD}_a(\alpha)$ (complements of the shaded regions). Finite family: $\alpha = \frac{25}{12}$ and $a = .4 < \frac{1}{2}$ with $c \in \{.03, .1, .2, .25, .3, .32768\}$ (left); traveling wave: $\alpha = \frac{4}{3}$ and $a = \frac{1}{2}$, with $c \in \{.05, .1, .2, .3, .4, .5\}$ (center); two phase: $\alpha = \frac{1}{2}$ and $a = 2 > \frac{1}{2}$, with $c \in \{.1, .4, .8, .98, 1, 1.02, 1.08\}$ (right).

Theorem 4.5.2. *Take $a > 0$ and $\alpha \in \mathbb{C} \setminus \{0\}$. There exists a simply connected domain Ω of conformal radius $c > 0$ containing 0 for which $\Omega \in QD_a(\alpha)$ if and only if $|\gamma| \geq 1$, with $\gamma = -\frac{a\overline{\alpha}}{c^{2a-1}}$ as in Theorem 4.5.1. In this case, Ω is unique modulo conformal radius, and $\Omega = \varphi(\mathbb{D}_*)$ with φ given by*

$$\varphi(z) = cz\frac{z - \overline{\gamma}^{\frac{1}{2a-1}}}{z - \gamma^{-\frac{1}{2a-1}}}\left(1 - \frac{\gamma^{\frac{1}{2a-1}}}{z}\right)^{\frac{1}{a}}, \qquad \gamma^{\frac{1}{2a-1}} = -\left(\frac{a\overline{\alpha}}{c^{2a-1}}\right)^{\frac{1}{2a-1}}. \tag{4.22}$$

Proof of Theorem 4.5.2. Let $0 \in \Omega \in \mathrm{QD}_a(h)$ be a simply connected PQD of conformal radius $c > 0$ with quadrature function $\alpha \in \mathbb{C} \setminus \{0\}$ for some $a > 0$. Then by Proposition 4.6.3,

$$\varphi(z) = zb_{z_0}(z)\left(|cz_0|^a\left(1 - \frac{\overline{z_0}^{-1}}{z}\right)\right)^{\frac{1}{a}}$$

is a Riemann map for Ω for some $z_0 \in \mathbb{D}_*$. Supposing that φ is univalent in $\mathbb{D}_*$, Equation 4.16 implies

$$\begin{aligned}
\alpha = h(w) &= \frac{1}{aw}\Phi_\varphi\left(C_{\mathbb{D}}\left[|cz_0|^a\left(1 - \frac{\overline{z_0}^{-1}}{z}\right)|cz_0|^a\left(1 - zz_0^{-1}\right)\right]\right)(w) - \frac{t}{w} \\
&= \frac{|cz_0|^{2a}}{aw}\Phi_\varphi\left(C_{\mathbb{D}}\left[1 + |z_0|^{-2} - zz_0^{-1} - \frac{\overline{z_0}^{-1}}{z}\right]\right)(w) - \frac{t}{w} \\
&= -\frac{z_0^{-1}|cz_0|^{2a}}{aw}c^{-1}w - \frac{\widetilde{t}}{w}.
\end{aligned}$$

hence $z_0^{-1}|z_0|^{2a} = -\frac{a\alpha}{c^{2a-1}} = -\overline{\gamma}$. This implies that $z_0\alpha < 0$, so we may conclude that $z_0 = \overline{\gamma}^{\frac{1}{2a-1}}$, and φ takes the form of Equation 4.22. In particular, whenever $\Omega \ni 0$ is a simply connected unbounded domain in $\mathrm{QD}_a(\alpha)$ for $a > 0$ and (wlog by change of variables) $\alpha > 0$, then $\Omega = \varphi(\mathbb{D}_*)$ with φ as above.

Regarding the univalence of 4.22, we apply Lemma 3.3.1, which tells us it is sufficient to check injectivity on the boundary. Suppose there existed $z \neq w \in \partial\mathbb{D}$ for which $\varphi(z) = \varphi(w)$. Then if we take $\eta = -\gamma^{\frac{1}{2a-1}}$,

$$|\varphi(z)| = c\left|\frac{z+\eta}{z+\eta^{-1}}\left(1+\frac{\eta^{-1}}{z}\right)^{\frac{1}{a}}\right| = c\eta\left|\left(1+\frac{\eta^{-1}}{z}\right)^{\frac{1}{a}}\right| = c\eta\left|z+\eta^{-1}\right|^{\frac{1}{a}},$$

so $\left|z-\eta^{-1}\right|^2 = \left|w-\eta^{-1}\right|^2$. Setting $z = e^{i\theta_0}$ and $w = e^{i\theta_2}$ and expanding, we find that $\cos(\theta_1) = \cos(\theta_2)$, so $w = \overline{z} = z^{-1}$. Thus

$$1 = \frac{\varphi(z^{-1})}{\varphi(z)} = cz^{-1}\frac{z^{-1}+\eta}{z^{-1}+\eta^{-1}}\left(1+z\eta^{-1}\right)^{\frac{1}{a}}\left(cz\frac{z+\eta}{z+\eta^{-1}}\left(1+\frac{\eta^{-1}}{z}\right)^{\frac{1}{a}}\right)^{-1} = \frac{\left(1+z^{-1}\eta^{-1}\right)^{2-\frac{1}{a}}}{\left(1+z\eta^{-1}\right)^{2-\frac{1}{a}}}.$$

Thus $(1+z^{-1}\eta^{-1})^{2-\frac{1}{a}} = (1+z\eta^{-1})^{2-\frac{1}{a}}$.

$a > \frac{1}{2}$ implies $2-\frac{1}{a} \in (0,2)$, in which case $z \mapsto z^{2-\frac{1}{a}}$ is injective in the closed right half-plane. Moreover, as long as $\eta \geq 1$, the images of $\partial\mathbb{D}$ under $z \mapsto 1+z\eta^{-1}$ and $z \mapsto 1+z^{-1}\eta^{-1}$ are in the closed right half plane. Thus $1+z\eta^{-1} = 1+z^{-1}\eta^{-1}$, so $z = \pm 1$. However recall that $w = z^{-1}$ which, in both cases, implies that $z = w$. Thus φ is univalent in $\mathbb{D}_*$ when $|\gamma| \geq 1$. On the other hand if $|\gamma| < 1$ then φ isn't even analytic in $\mathbb{D}_*$. In particular, φ is univalent in $\mathbb{D}_*$ if and only if $|\gamma| \geq 1$. □

We saw in Theorems 4.5.1 and 4.5.2 that the family of simply connected Ω in $\mathrm{QD}_a(\alpha)$ depends crucially on the value of a. Another way of understanding this is through Propositions 4.4.1 and 4.4.2 which tell us $\Omega \in \mathrm{QD}_a(\alpha)$ if and only if $K := \Omega^c$ is a local droplet of the potential $Q(w) = \frac{|w|^{2a}}{a^2} - 2\mathrm{Re}(\alpha w)$. So if K is such a local droplet then, by Lemmas 2.2.2 and 2.2.3, there exists an associated Hele-Shaw chain $\{K_t\}_{0<t\leq t_0}$, where $K_{t_0} = K$ such that each element of this chain contains a local minimum of Q. Thus $\mathrm{QD}_a(\alpha)$ is nonempty only when Q has a local minimum. For example, in Figure 4.4, we observe that the families of solutions each "nucleate" at the unique local minimum of Q, located at 0, 0, and $(a\alpha)^{\frac{1}{2a-1}}$ respectively.

Classification of Monomial PQDs

In this section, we are interested in characterizing PQDs with a monomial quadrature function - that is, $\Omega \in \mathrm{QD}_a(\alpha k w^{k-1})$ for some $a > 0$, $\alpha \in \mathbb{C}\setminus\{0\}$, and $k \in \mathbb{Z}_+$. Firstly note that we can wlog assume that $\alpha > 0$ because if S_a is the associated generalized Schwarz function then

$$\widetilde{S_a}(w) := \left(\frac{|\alpha|}{\alpha}\right)^{\frac{1}{k}} S_a\left(w\left(\frac{|\alpha|}{\alpha}\right)^{\frac{1}{k}}\right) \doteq \left(\frac{|\alpha|}{\alpha}\right)^{\frac{1}{k}}\frac{1}{a}\overline{w\left(\frac{|\alpha|}{\alpha}\right)^{\frac{1}{k}}}\left|w\left(\frac{|\alpha|}{\alpha}\right)^{\frac{1}{k}}\right|^{2(a-1)} = \frac{1}{a}\overline{w}|w|^{2(a-1)}$$

is a generalized Schwarz function for $\widetilde{\Omega} := \left(\frac{\alpha}{|\alpha|}\right)^{\frac{1}{k}}\Omega$, so it is a PQD. It is then straightforward to verify that $\widetilde{\Omega} \in \mathrm{QD}_a(|\alpha|kw^{k-1})$.

Theorem 4.5.3. *Fix $a > 0$, $\alpha \in \mathbb{C} \setminus \{0\}$, $k \in \mathbb{Z}_+$. If $0 \notin \Omega$ is a $\mathbb{Z}_k$−rotationally symmetric domain ($e^{\frac{2\pi i}{k}}\Omega = \Omega$), then $\Omega \in QD_a(\alpha k w^{k-1})$ if and only if $\Omega^k \in QD_{\frac{a}{k}}(\alpha k^2)$. In this case, $\Omega = \widetilde{\varphi}(\mathbb{D}_*)$, where*

$$\widetilde{\varphi}(z) = cz\left(1 - \frac{\gamma_k}{z^k}\right)^{\frac{1}{a}}, \quad \gamma_k = -\frac{a\overline{\alpha}k}{c^{2a-k}}. \tag{4.23}$$

is univalent in $\mathbb{D}_*$.

Toward proving this, we note the following relationship between the $k = 1$ case and the general case:

Lemma 4.5.1. Fix $a > 0$, $\alpha \in \mathbb{C} \setminus \{0\}$, and $k \in \mathbb{Z}_+$. If $\Omega \in \mathrm{QD}_{\frac{a}{k}}(\alpha k^2)$ then $\{w : w^k \in \Omega\} \in \mathrm{QD}_a\left(\alpha k w^{k-1}\right)$.

Proof of Lemma 4.5.1. Suppose that $\Omega \in \mathrm{QD}_{\frac{a}{k}}(\alpha)$ and define $\widetilde{\Omega} := \{w : w^k \in \Omega\}$. By Theorem 4.2.1, Ω admits a generalized Schwarz function $S_{\frac{a}{k}}(w) \doteq \frac{k}{a}\overline{w}|w|^{2\left(\frac{a}{k}-1\right)}$, so

$$S_a(w) := \frac{1}{k}w^{k-1}S_{\frac{a}{k}}(w^k) \doteq \frac{1}{k}w^{k-1}\frac{k}{a}\overline{w}^k|w|^{2(a-k)} = \frac{1}{a}\overline{w}|w|^{2(a-1)}$$

is a generalized Schwarz function for $\widetilde{\Omega}$. Thus, by Theorem 4.2.1, $\widetilde{\Omega} \in \mathrm{QD}_a$ and

$$\widetilde{h}(w) = C_{\widetilde{\Omega}_*}\left[S_a(w)\right] = C_{\widetilde{\Omega}_*}\left[\frac{1}{k}w^{k-1}S_{\frac{a}{k}}(w^k)\right] = C_{\widetilde{\Omega}_*}\left[\frac{\alpha}{k}w^{k-1} - \frac{1}{k}w^{k-1}G(w^k)\right] = \frac{\alpha}{k}w^{k-1}.$$

The result follows from the substitution $\alpha \mapsto \alpha k^2$. □

It is not true however that every element of $\mathrm{QD}_a(\alpha k w^{k-1})$ can be represented this way. To wit,

Example 4.5.1. Consider the map $\varphi : \mathbb{D}_* \to \widehat{\mathbb{C}}$ given by

$$\varphi(z) := cz\frac{z+u}{z+u^{-1}}\sqrt{\left(1 - \frac{2\gamma}{u^3}z^{-1}\right)\left(1 + \frac{1}{u}z^{-1}\right)} = c\left(z+u\right)\sqrt{\frac{z - 2\gamma u^{-3}}{z + u^{-1}}}$$

where $\gamma = -\frac{2\alpha}{c^2}$, $u = \sqrt{\sqrt{\gamma\left(\gamma+4\right)} - \gamma}$, and $c, \alpha > 0$. One can show that this map is univalent in $\mathbb{D}_*$ precisely when $\frac{2\alpha}{c^2} = -\gamma \geq \frac{5\sqrt{13}-1}{4} \approx 4.257$ via Lemma 3.3.1 and solving the equation $\varphi^2(z) = \varphi^2(\xi)$ over $|z| = |\xi| = 1$ (taking its conjugate, we obtain $\varphi^2(z^{-1}) = \varphi^2(\xi^{-1})$, so we may eliminate ξ to obtain a polynomial equation in z with coefficients depending only on γ).

Then note that $\varphi_{\text{out}}^2(z) = r^\#(z) = u^2c^2\left(1 - \frac{2\gamma}{u^3}z^{-1}\right)\left(1 + \frac{1}{u}z^{-1}\right)$ has the form required by Theorem 4.3.3, so $\Omega := \varphi(\mathbb{D}_*)$ is a PQD when φ is univalent. Applying Equation 4.11 with $r(z) = u^2c^2(1 - 2\gamma u^{-3}z)(1 + u^{-1}z)$ we obtain

$$\begin{aligned} h(w) &= \frac{1}{2}C_{\Omega_*}\left[\frac{(rr^\#)\circ\psi(w)}{w}\right] = \frac{c^4}{2}C_{\Omega_*}\left[\frac{-2\gamma\psi^2(w) + u^{-3}(u^2 - 2\gamma)(u^4 - 2\gamma)\psi(w) + O(1)}{w}\right] \\ &= C_{\Omega_*}\left[-c^2\gamma w + \frac{c^3}{2u}\left(u^4 + 2u^2\gamma - 4\gamma\right) + O(w^{-1})\right] \\ &= 2\alpha w + \frac{c^3}{2u}\left(u^4 + 2u^2\gamma - 4\gamma\right) \end{aligned}$$

And u, as defined above, is a root of $u^4 + 2\gamma u^2 - 4\gamma$, so $h(w) = 2\alpha w$. That is, $\Omega \in \Omega_2\left(2\alpha w\right)$. See Figure 4.5. It is of note that $-\Omega$ is also in $\mathrm{QD}_2\left(2\alpha w\right)$ (consider taking $c \mapsto -c$).

Figure 4.5: An asymmetric family (complements of the shaded regions) in $\mathrm{QD}_2\left(\frac{w}{2}\right)$ for $c \in \{.09, .2, .27, .31, 0.34272\}$ (Example 4.5.1)

Despite the failure of the converse of Lemma 4.5.1 in the general case, it can nevertheless be recovered with the additional requirement of $\mathbb{Z}_k-$rotational symmetry.

Lemma 4.5.2 (Converse to Lemma 4.5.1)**.** Fix $a > 0$, $\alpha \in \mathbb{C} \setminus \{0\}$, and $k \in \mathbb{Z}_+$. If $\Omega \in \mathrm{QD}_a(\alpha k w^{k-1})$ and is $\mathbb{Z}_k-$rotationally symmetric ($e^{\frac{2\pi i}{k}}\Omega = \Omega$), then $\Omega^k \in \mathrm{QD}_{\frac{a}{k}}(\alpha k^2)$.

Proof of Lemma 4.5.2. Let Ω be an unbounded domain as above so that for each $f \in \mathcal{A}_0(\Omega^k)$,

$f(w^k) \in \mathcal{A}_0(\Omega)$, and

$$\begin{aligned}\int_{\Omega^k} f(w)|w|^{2\left(\frac{a}{k}-1\right)}dA(w) &= \int_{\Omega} f(w^k)k^2|w|^{2(a-1)}dA(w) \\ &= \oint_{\partial\Omega} f(w^k)\alpha k^3 w^{k-1}dw \\ &= \oint_{\partial\Omega^k} f(w)\alpha k^2 dw.\end{aligned}$$

Thus $\Omega^k \in \mathrm{QD}_{\frac{a}{k}}(\alpha)$. The bounded case is completely analogous. $\square$

We're now equipped to prove Theorem 4.5.3.

Proof of Theorem 4.5.3. Let Ω be a $\mathbb{Z}_k$−rotationally symmetric domain. If $\Omega \in \mathrm{QD}_a(\alpha k w^{k-1})$ then by Lemma 4.5.2, $\Omega^k \in \mathrm{QD}_{\frac{a}{k}}(\alpha k^2)$. On the other hand, if $\Omega^k \in \mathrm{QD}_{\frac{a}{k}}(\alpha k^2)$ then by Lemma 4.5.1, $\Omega = \{w : w^k \in \Omega^k\}$ is in $\mathrm{QD}_a(\alpha k w^{k-1})$.

Now, if φ is the Riemann map associated to Ω^k and $0 \notin \Omega$ (and thus also $0 \notin \Omega^k$) then by Equation 4.21, there exists a $c > 0$ for which $\varphi(z) = c^k z\left(1 - \gamma_k z^{-1}\right)^{\frac{k}{a}}$, where $\gamma_k = -\frac{\left(\frac{a}{k}\right)\overline{(\alpha k^2)}}{(c^k)^{2\frac{a}{k}-1}} = -\frac{a\overline{\alpha}k}{c^{2a-k}}$. Moreover, by Proposition 4.6.3, there exists a polynomial p of degree k with $p(0) = 1$ for which $\widetilde{\varphi}(z) = c' z p^{\#}(z)^{\frac{1}{a}}$. Thus,

$$(c')^k z^k p^{\#}(z)^{\frac{k}{a}} = \left(c' z p^{\#}(z)^{\frac{1}{a}}\right)^k = \widetilde{\varphi}^k(z) = \varphi(z^k) = c^k z^k\left(1 - \gamma_k z^{-k}\right)^{\frac{k}{a}}.$$

So, $(c')^k p^{\#}(z)^{\frac{k}{a}} = c^k\left(1 - \gamma_k z^{-k}\right)^{\frac{k}{a}}$. Taking $z \to \infty$, we find that $c' = c$, and so $p^{\#}(z)^{\frac{1}{a}} = \left(1 - \gamma_k z^{-k}\right)^{\frac{1}{a}}$. The result follows. $\square$

Figure 4.4 exhibits each of the cases/phenomena described in Theorems 4.5.1 and 4.5.2. Furthermore, by exploiting Lemma 4.5.1 and Equation 4.23, we obtain explicit families $\{\Omega_t\}_t \subseteq \mathrm{QD}_a(\alpha k w^{k-1})$, exhibited in Figures 4.6 and 4.7.

4.6 Further Examples

One-Point PQDs of Higher Order

Let $X_k(w_0)$ be the space of rational functions $= 0$ at ∞ with a unique pole of order $k \in \mathbb{Z}_+$ at $w_0 \in \mathbb{C}$.[3]

Proposition 4.6.1. Fix $a > 0$ and let Ω be a bounded, simply connected domain with Riemann map $\varphi : \mathbb{D} \to \Omega$ such that $\varphi(0) = w_0$, and $\varphi'(0) > 0$.

[3] i.e. $\sum_{j=1}^{k} \frac{\alpha_j}{(w-w_0)^j}$ for some collection of $\alpha_j \in \mathbb{C}$.

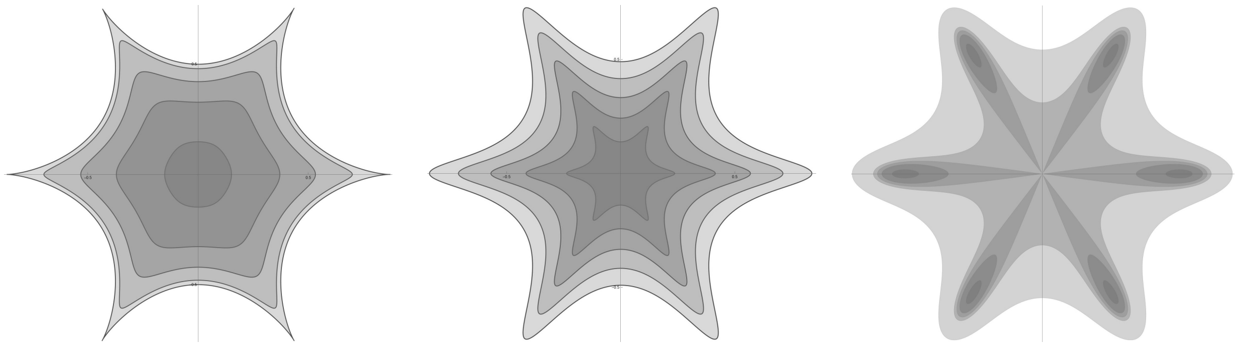

Figure 4.6: Families $\{\Omega_t\}_t$ of PQDs (complements of the shaded regions) in $\mathrm{QD}_2\left(\frac{w^{k-1}}{2}\right)$, $\mathrm{QD}_3\left(\frac{2}{9}w^{k-1}\right)$, and $\mathrm{QD}_4\left(\frac{w^{k-1}}{8}\right)$ with $k = 6$ (left to right). When $a = 2$, $k > 2a$, so the family has a maximal element. When $a = 3$, $k = 2a$, so this is the "traveling wave" case. Finally, when $a = 4$, $k < 2a$, so we obtain a two-phase family which is initially multiply connected and is simply connected past the critical time.

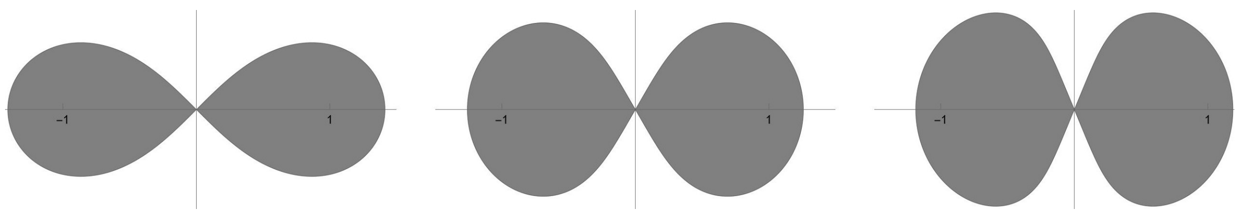

Figure 4.7: $\Omega \in \mathrm{QD}_2\left(\frac{w}{2}\right)$, $\mathrm{QD}_3\left(\frac{2}{9}w\right)$, and $\mathrm{QD}_\alpha\left(\frac{w}{8}\right)$ (left to right, complements of the shaded regions). The interior angles at 0 are $\frac{\pi}{a}$ radians.

1. If $0 \notin \Omega$, then there exists $h \in X_k(w_0)$ such that $\Omega \in \mathrm{QD}_a(h)$ if and only if there exists a polynomial p of degree k such that

$$\varphi(z) = p(z)^{\frac{1}{a}}. \tag{4.24}$$

2. If $w_0 = 0$ (so $0 \in \Omega$), then there exists $h \in X_k(0)$ such that $\Omega \in \mathrm{QD}_a(h)$ if and only if there exists a polynomial p of degree $k - 1$ such that

$$\varphi(z) = zp(z)^{\frac{1}{a}}. \tag{4.25}$$

3. If $0 \in \Omega$ and $w_0 \neq 0$, then there exists $h \in X_k(w_0)$ such that $\Omega \in \mathrm{QD}_a(h)$ if and only if there exists a polynomial p of degree k with a root at $\overline{z_0}^{-1}$ such that

$$\varphi(z) = b_{z_0}(z)p(z)^{\frac{1}{a}}. \tag{4.26}$$

Proof of Proposition 4.6.1.
We will only carry out the proof of case (1) because cases (2) and (3) follow by similar arguments.

Let $\Omega \not\ni 0$ be a bounded domain and $\varphi : \mathbb{D} \to \Omega$ a Riemann map for Ω for which $\varphi(0) = w_0$ and $\varphi'(0) > 0$.

Suppose there exists $h \in X_k(w_0)$ such that $\Omega \in \mathrm{QD}_a(h)$. Then by Theorem 4.3.4 and Corollary 4.3.2, there exists a polynomial q of degree $k-1$ for which $\varphi(z) = \left(w_0^a + zq(z)\right)^{\frac{1}{a}}$. Thus there exists a polynomial p of degree k for which $\varphi(z) = p(z)^{\frac{1}{a}}$.

Conversely, suppose there exists a polynomial p of degree k such that $\varphi(z) = p(z)^{\frac{1}{a}}$. Then by Theorem 4.3.3, $\Omega \in \mathrm{QD}_a(h)$ for some rational h. Moreover by Theorem 4.3.5,

$$\begin{aligned} h(w) &= \frac{1}{aw}\Phi_\varphi\left(C_{\mathbb{D}_*}\left[pp^\#\right]\right)(w) + \frac{t}{w} = \frac{1}{aw}\Phi_\varphi\left(C_{\mathbb{D}_*}\left[\sum_{|j|\le k}\beta_j z^j\right]\right)(w) + \frac{t}{w} \\ &= \frac{1}{aw}\Phi_\varphi\left(\sum_{j=1}^{k}\beta_{-j}z^{-j}\right)(w) + \frac{t}{w} \\ &= \frac{1}{w}\sum_{j=1}^{k}\frac{\gamma_j}{(w-\varphi(0))^j} + \frac{t}{w} = \sum_{j=1}^{k}\frac{\widetilde{\gamma}_j}{(w-w_0)^j} + \frac{\widetilde{t}}{w}. \end{aligned}$$

But then because $h \in \mathcal{A}_0(\Omega_*)$, we find that $\widetilde{t} = 0$, so $h \in X_k(w_0)$. □

Proposition 4.6.2. Take $a > 0$ and Ω an unbounded, simply connected domain with conformal radius c admitting a Riemann map $\varphi : \mathbb{D}_* \to \Omega$ with $\varphi'(\infty) = c$, and $X_k(w_0)$ the space of rational functions $= 0$ at ∞ with a unique pole of order $k \in \mathbb{Z}_+$ at $w_0 \in \Omega$. Then

1. If $0 \notin \Omega$, then there exists $h \in X_k(w_0)$ such that $\Omega \in \mathrm{QD}_a(h)$ if and only if there exists $z_0 \in \mathbb{D}_*$ with $\varphi(z_0) = w_0$, and a polynomial p of degree k with $p(0) = c^a$ and without a root at z_0 such that
$$\varphi(z) = z\left(\frac{p^\#(z)}{\left(1 - \frac{\overline{z_0}^{-1}}{z}\right)^k}\right)^{\frac{1}{a}}. \tag{4.27}$$

2. If $0 \in \Omega$ then there exists $h \in X_k(w_0)$ such that $\Omega \in \mathrm{QD}_a(h)$ if and only if there exist $z_0, z_1 \in \mathbb{D}_*$ with $\varphi(z_0) = w_0$, $\varphi(z_1) = 0$, and a polynomial p of degree k with $p(0) = |cz_1|^a$ a root at z_1 and without a root at z_0 such that
$$\varphi(z) = zb_{z_1}(z)\left(\frac{p^\#(z)}{\left(1 - \frac{\overline{z_0}^{-1}}{z}\right)^k}\right)^{\frac{1}{a}}. \tag{4.28}$$

Proof of Proposition 4.6.2.

Take $a > 0$ and Ω an unbounded, simply connected domain with conformal radius c admitting a

Riemann map $\varphi : \mathbb{D}_* \to \Omega$ with $\varphi'(\infty) = c$.

(**1**) **Forward direction:**

Suppose $0 \notin \Omega$ and there exists $h \in X_k(w_0)$ such that $\Omega \in \mathrm{QD}_a(h)$. Then by Theorem 4.3.6 and corollary 4.3.3,

$$\varphi(z) = z\left(c^a + \frac{1}{z}\sum_{j=1}^{k}\frac{\beta_j}{(z^{-1} - \overline{z_0})^j}\right)^{\frac{1}{a}} = z\left(\frac{c^a q(z)}{\left(z - \overline{z_0}^{-1}\right)^k}\right)^{\frac{1}{a}},$$

where $\varphi(z_0) = w_0$, and q is a monic polynomial of degree k with $q(\overline{z_0}^{-1}) \neq 0$. Thus there exists a polynomial p of degree k with $p(z_0) \neq 0$, and $p(0) = c^a$, for which $\varphi(z) = z\left(\frac{p^{\#}(z)}{\left(1 - \frac{\overline{z_0}^{-1}}{z}\right)^k}\right)^{\frac{1}{a}}$.

(**1**) **Backward direction:**

Conversely, suppose there exists $z_0 \in \mathbb{D}_*$ with $\varphi(z_0) = w_0$, and a polynomial p of degree k with $p(0) = c^a$ and without a root at z_0 such that $\varphi(z) = z\left(\frac{p^{\#}(z)}{\left(1 - \frac{\overline{z_0}^{-1}}{z}\right)^k}\right)^{\frac{1}{a}}$. Then, by Theorem 4.3.7,

$$\begin{aligned} h(w) &= \frac{1}{aw}\Phi_\varphi\left(C_{\mathbb{D}}\left[\frac{p^{\#}(z)}{\left(1 - \frac{\overline{z_0}^{-1}}{z}\right)^k}\frac{p(z)}{\left(1 - zz_0^{-1}\right)^k}\right]\right)(w) - \frac{t}{w} \\ &= \frac{1}{aw}\Phi_\varphi\left(C_{\mathbb{D}}\left[\frac{z_0^k z^k p^{\#}(z)}{\left(z - \overline{z_0}^{-1}\right)^k}\frac{p(z)}{(z_0 - z)^k}\right]\right)(w) - \frac{t}{w}. \end{aligned}$$

Neither p nor $p^{\#}$ have a root at z_0, so the argument of Φ_φ is meromorphic in $\mathbb{D}_*$ and has a unique pole of order k at z_0 in $\mathbb{D}_*$. Hence, by partial fraction decomposition,

$$h(w) = \frac{1}{aw}\sum_{j=1}^{k}\frac{\beta_j}{(w - \psi(z_0))^j} + \frac{t}{w} = \sum_{j=1}^{k}\frac{\widetilde{\beta_j}}{(w - w_0)^j} + \frac{\widetilde{t}}{w}.$$

Finally $0 \in \Omega_*$ but $h \in \mathcal{A}(\Omega_*)$, so $\widetilde{t} = 0$, and we conclude that $h \in X_k(w_0)$.

The argument is similar for case (**2**). □

Polynomial PQDs

Let Ω be a simply connected unbounded domain not containing zero with conformal radius $c > 0$. Now take $a > 0$, $k \in \mathbb{Z}_{\geq 0}$ and suppose $\Omega \in \mathrm{QD}_a(h)$ for some polynomial h of degree k.

Then by Theorem 4.3.7 and corollary 4.3.3, there exists a polynomial p of degree k for which $\varphi(z) = z\left(c^a + z^{-1}p^{\#}(z)\right)^{\frac{1}{a}}$.

Conversely, suppose $0 \notin \Omega$ and there exists a polynomial p of degree k for which $\varphi(z) = z\left(c^a + z^{-1}p^{\#}(z)\right)^{\frac{1}{a}}$. Then by Equation 4.16,

$$awh(w) = \Phi_\varphi\left(C_{\mathbb{D}}\left[rr^{\#}\right]\right)(w) - at = \Phi_\varphi\left(C_{\mathbb{D}}\left[\left(c^a + z^{-1}p^{\#}(z)\right)(c^a + zp(z))\right]\right)(w) - at$$

The argument of the Faber transform is a Laurent polynomial of the form $\sum_{|j|\le k+1}\beta_j z^j$, so its analytic part in $\mathbb{D}$ is a polynomial of degree $k+1$. The Faber transform of a polynomial of degree $k+1$ is a polynomial of degree $k+1$, so we find that $h(w) = \frac{\beta}{w} + q(w)$ for some polynomial q of degree k. Finally, $h \in \mathcal{A}(\Omega_*)$ implies $\beta = 0$ when $0 \notin \Omega$. Thus h is a polynomial of degree k. This proves the first case of the following proposition.

Proposition 4.6.3. Let Ω be an unbounded, simply connected domain with conformal radius c, admitting a Riemann map $\varphi : \mathbb{D}_* \to \Omega$. Then for each $a > 0$ and $k \in \mathbb{Z}_{\ge 0}$,

1. If $0 \notin \Omega$, then there exists a polynomial h of degree k such that $\Omega \in \mathrm{QD}_a(h)$ if and only if there exists a polynomial p of degree $k+1$ with $p(0) = c^a$ such that
$$\varphi(z) = zp^{\#}(z)^{\frac{1}{a}}. \tag{4.29}$$
2. If $0 \in \Omega$, then there exists a polynomial h of degree k such that $\Omega \in \mathrm{QD}_a(h)$ if and only if there exists a polynomial p of degree $k+1$ with $p(0) = |cz_0|^a$ and a root at z_0 for which
$$\varphi(z) = zb_{z_0}(z)p^{\#}(z)^{\frac{1}{a}}. \tag{4.30}$$

(recall that $b_\lambda(z) = \frac{\bar\lambda}{|\lambda|}\frac{z-\lambda}{z\bar\lambda - 1}$ is a Blaschke factor). Note that, by applying Proposition 4.4.3 to case (1) for $k \le 2a$, we find that φ is univalent in $\mathbb{D}_*$ if and only if each of the roots of p are contained in $\mathbb{D}^c$. In particular, the set of unbounded PQDs not containing zero with polynomial quadrature function h of degree $k \le 2a$ is parameterized by the set of polynomials p of degree $k+1$ for which $z \mapsto zp^{\#}(z)^{\frac{1}{a}}$ is univalent in $\mathbb{D}_*$. The proof of case 2 is given below.

Proof of Proposition 4.6.3.
(**2**) **Forward direction:** Let $0 \in \Omega \in \mathrm{QD}_a(h)$ be a simply connected and unbounded domain of conformal radius $c > 0$ for some polynomial h of degree $k \in \mathbb{Z}_{\ge 0}$ and $a > 0$. Then by Equation 4.16 and corollary 4.3.3, there exists a polynomial p of degree $k+1$ with $p(0) = |cz_0|^a$ for which $\varphi(z) = zb_{z_0}(z)p^{\#}(z)^{\frac{1}{a}}$ is a Riemann map for Ω. Then by Equation 4.11, we have that

$$h(w) = \frac{1}{a}C_{\Omega_*}\left[\frac{\left(pp^{\#}\right)\circ\psi(w)}{w}\right] = \frac{1}{a}C_{\Omega_*}\left[\frac{p\circ\psi(w)}{w}p^{\#}\circ\psi(w)\right]$$

Note that $p^\#$ has no roots in $\mathbb{D}_*$ (otherwise φ wouldn't be analytic in $\mathbb{D}_*$), so if $p(z_0) \neq 0$, the argument of the RHS must have a pole at zero, which would imply h has a pole at 0, a contradiction. Thus $p(z_0) = 0$.

(**2**) **Backward direction:** On the other hand, if there exists a polynomial p of degree $k+1$ with a root at z_0 for which $\varphi(z) = zb_{z_0}(z)p^\#(z)^{\frac{1}{a}}$ then by Theorem 4.3.3, $\Omega \in \mathrm{QD}_a(h)$ for some rational h. We saw above that Equation 4.11 tells us

$$h(w) = \frac{1}{a}C_{\Omega_*}\left[\frac{p\circ\psi(w)}{w}p^\#\circ\psi(w)\right],$$

the argument of which is meromorphic in Ω and has no poles in Ω, except a pole of order k at ∞. Thus h is a polynomial of degree k. $\square$

Example 4.6.1. $\varphi(z) = z\left(1+\frac{1}{2}z^{-1}+\frac{3}{4}z^{-2}\right)^{\frac{1}{2}}$, is univalent in $\mathbb{D}_*$ by Proposition 4.4.3 because its roots, $\frac{-1\pm\sqrt{11}i}{4}$ are contained in $\mathbb{D}$. Then $\Omega := \varphi(\mathbb{D}_*)$ is an unbounded PQD in $\mathrm{QD}_2\left(\frac{3}{8}w+\frac{1}{4}\right)$. To see this, first note that if $\varphi(z) = 0$ then $|z| = \frac{\sqrt{3}}{2} < 1$, so $0 \notin \Omega$. Then by Proposition 4.6.3, $\Omega \in \mathrm{QD}_2(aw+b)$ and Equation 4.11 implies

$$\begin{aligned} aw+b &= C_{\Omega_*}\left[\frac{(pp^\#)\circ\psi(w)}{2w}\right] \\ &= C_{\Omega_*}\left[\frac{\left(\frac{3}{4}z^2+\frac{7}{8}z+O(1)\right)\circ\psi(w)}{2w}\right] \\ &= C_{\Omega_*}\left[\frac{\frac{3}{4}\psi^2(w)+\frac{7}{8}\psi(w)}{2w}\right] \\ &= \frac{3}{8}w+\frac{1}{4}. \end{aligned}$$

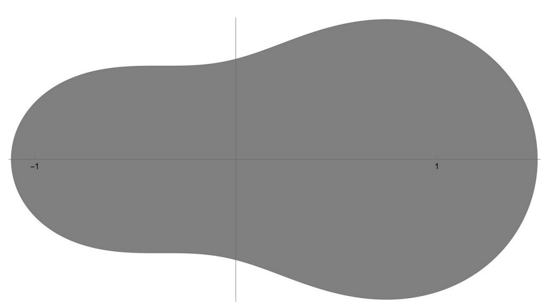

Figure 4.8: $\Omega \in \mathrm{QD}_2\left(\frac{3}{8}w+\frac{1}{4}\right)$ (complement of shaded region).

Linear PQDs

By Proposition 4.6.3, if $\Omega \in \mathrm{QD}_a(\alpha_0 + \alpha_1 w)$ is simply connected and does not contain 0, then there exists a polynomial p of degree 2 with $p(0) = c^a$ for which the associated Riemann map $\varphi : \mathbb{D}_* \to \Omega$ is given by $\varphi(z) = zp^\#(z)^{\frac{1}{a}}$. That is, there exist constants $c_1, c_2 \in \mathbb{C}$ for which $p(z) = c^a(1 + c_1 z + c_2 z^2)$. By change of variables we can wlog assume $\alpha_1 = 1$. Applying Theorem 4.3.7, we have that

$$\begin{aligned}
\alpha_0 + w &= \frac{1}{aw}\Phi_\varphi\left(\mathcal{C}_{\mathbb{D}}\left[pp^\#\right]\right)(w) - \frac{t}{w} \\
&= \frac{c^{2a}}{aw}\Phi_\varphi\left(u_0 + u_1 z + c_2 z^2\right)(w) - \frac{t}{w} \\
&= \frac{c^{2a}}{aw}\left(u_0 + u_1 F_1(w) + c_2 F_2(w)\right) - \frac{t}{w},
\end{aligned}$$

where $u_0 = 1 + |c_1|^2 + |c_2|^2$ and $u_1 = c_1 + \overline{c_1}c_2$. Using Equation 2.11 we compute $F_1(w) = \frac{w}{c} - \frac{\overline{c_1}}{a}$ and $F_2(w) = \frac{w^2}{c^2} - w\frac{2\overline{c_1}}{ac} + \frac{\overline{c_1}^2 - 2\overline{c_2}}{a}$. Substituting and equating polynomial coefficients, we obtain $1 = \frac{c^{2(a-1)}}{a}c_2$, $\alpha_0 = c_1\frac{c^{2a-1}}{a} + \frac{a-2}{a}c\overline{c_1}$. Solving for c_1 and c_2 yields

$$\begin{aligned}
c_1 &= \frac{a}{c}\frac{(a-2)\overline{\alpha_0} - c^{2(a-1)}\alpha_0}{(a-2)^2 - c^{4(a-1)}}, \\
c_2 &= \frac{a}{c^{2(a-1)}}.
\end{aligned}$$

In particular, we have shown:

Proposition 4.6.4. If $\Omega \in \mathrm{QD}_a(\alpha_0 + \alpha_1 w)$ is a simply connected domain not containing zero, then

$$\varphi(z) = cz\left(1 + \frac{a}{c}\frac{(a-2)\alpha_0 - c^{2(a-1)}\overline{\alpha_0}}{(a-2)^2 - c^{4(a-1)}}z^{-1} + \frac{a}{c^{2(a-1)}}z^{-2}\right)^{\frac{1}{a}}$$

is a Riemann map for φ, where $c = \varphi'(\infty) > 0$ is the conformal radius of Ω.

Note that if we set $a = 2$ and $\alpha_0 = 2\sqrt{2}$, then this simplifies to

$$\varphi(z) = cz\left(1 + \frac{4\sqrt{2}}{c^3}z^{-1} + \frac{2}{c^2}z^{-2}\right)^{\frac{1}{2}}.$$

Furthermore, $0 \notin \Omega = \varphi(\mathbb{D}_*)$ for all $c > \sqrt{2}$ and in the limit as $c \to \sqrt{2}$, the expression simplifies further to $\varphi(z) = \sqrt{2}(z+1)$. In particular, this suggests that $\mathbb{D}_{\sqrt{2}}(\sqrt{2})_* \in \mathrm{QD}_2\left(2\sqrt{2} + w\right)$. However, $0 \in \mathrm{Cl}(\mathbb{D}_{\sqrt{2}}(\sqrt{2}))$, so this isn't fully rigorous. It also breaks the observed pattern of $\partial\Omega$ forming angles of $\frac{\pi}{a}$ radians. To confirm that $\mathbb{D}_{\sqrt{2}}(\sqrt{2})_* \in \mathrm{QD}_2\left(2\sqrt{2} + w\right)$, we will verify the quadrature identity directly. Take $f \in \mathcal{A}_0(\mathbb{D}_{\sqrt{2}}(\sqrt{2}))$ so that, by Green's theorem:

$$\int_{\mathbb{D}_{\sqrt{2}}(\sqrt{2})_*} f(w)|w|^2 dA(w) = \frac{1}{2}\oint_{\partial\mathbb{D}_{\sqrt{2}}(\sqrt{2})_*} f(w)w\overline{w}^2 dw$$

Applying the identity $\overline{w} \doteq \frac{2}{w-\sqrt{2}} + \sqrt{2}$ on $\partial\mathbb{D}_{\sqrt{2}}(\sqrt{2})$ and expanding, we obtain:

$$\begin{aligned}
\int_{\mathbb{D}_{\sqrt{2}}(\sqrt{2})_*} f(w)|w|^2 dA(w) &= \frac{1}{2}\oint_{\partial\mathbb{D}_{\sqrt{2}}(\sqrt{2})_*} f(w)w\left(\frac{2}{w-\sqrt{2}} + \sqrt{2}\right)^2 dw \\
&= \oint_{\partial\mathbb{D}_{\sqrt{2}}(\sqrt{2})_*} f(w)\left(\frac{2\sqrt{2}}{(w-\sqrt{2})^2} + \frac{6}{w-\sqrt{2}} + 2\sqrt{2} + w\right) dw \\
&= \oint_{\partial\mathbb{D}_{\sqrt{2}}(\sqrt{2})_*} f(w)\left(2\sqrt{2} + w\right) dw,
\end{aligned}$$

confirming the identity.

Quadratic PQDs

By Proposition 4.6.3, if $\Omega \in \mathrm{QD}_a(\alpha_0 + \alpha_1 w + \alpha_2 w^2)$ is simply connected and does not contain 0, then there exists a polynomial p of degree 3 with $p(0) = c^a$ for which the associated Riemann map $\varphi : \mathbb{D}_* \to \Omega$ is given by $\varphi(z) = zp^\#(z)^{\frac{1}{a}}$. That is, there exist constants $c_1, c_2, c_3 \in \mathbb{C}$ for which $p(z) = c^a(1 + c_1 z + c_2 z^2 + c_3 z^3)$. By change of variables we can wlog assume $\alpha_2 = 1$. Applying Theorem 4.3.7, we have that

$$\begin{aligned}
\alpha_0 + \alpha_1 w + w^2 &= \frac{1}{aw}\Phi_\varphi\left(C_{\mathbb{D}}\left[pp^\#\right]\right)(w) - \frac{t}{w} \\
&= \frac{c^{2a}}{aw}\Phi_\varphi\left(u_0 + u_1 z + u_2 z^2 + c_3 z^3\right)(w) - \frac{t}{w} \\
&= \frac{c^{2a}}{aw}\left(u_0 + u_1 F_1(w) + u_2 F_2(w) + c_3 F_3(w)\right) - \frac{t}{w},
\end{aligned}$$

Where $u_0 = 1 + |c_1|^2 + |c_2|^2 + |c_3|^2$, $u_1 = c_1 + \overline{c_1}c_2 + \overline{c_2}c_3$, $u_2 = c_2 + \overline{c_1}c_3$. Computing F_1, F_2, and F_3 explicitly (using e.g. Equation 2.11) and equating coefficients, we find

$$\begin{aligned}
\alpha_0 &= \frac{c^{2a-1}}{2a^3}\left(2a^2 c_1 + 2a(a-2)\overline{c_1}c_2 + (a-3)(2a\overline{c_2} - \overline{c_1}^2)c_3\right) \\
\alpha_1 &= \frac{c^{2a-2}}{a^2}\left(ac_2 + (a-3)\overline{c_1}c_3\right) \\
1 &= \frac{c^{2a-3}}{a}c_3
\end{aligned}$$

We can then write c_3 in terms of a and c:

$$c_3 = ac^{3-2a}$$

and we can write c_2 in terms of a, c, and c_1:

$$c_2 = a\alpha_1 c^{2(1-a)} - (a-3)c^{3-2a}\overline{c_1}.$$

Substituting these into the α_0 equation, the problem reduces to that of solving an algebraic equation in c_1 (linear in c_1 and quadratic in $\overline{c_1}$) with coefficients depending only on a, c, α_0, and α_1:

$$\begin{aligned} 0 &= \left((a-3)(2a-3)c^{2a+3}\right)\overline{c_1}^2 - \left(2(a-2)ac^{2a+2}\alpha_1\right)\overline{c_1} \\ &\quad + 2a\left(((a-3)^2c^6 - c^{4a})c_1 + ac^{2a+1}\alpha_0 - (a-3)ac^5\overline{\alpha_1}\right). \end{aligned}$$

QD$_2(\alpha_0 + \alpha_1 w + w^2)$

When $a = 2$, the logarithmic potential associated to a quadratic PQD Ω (Proposition 4.4.2) is of the form

$$Q(w) = \frac{|w|^4}{4} - 2\mathrm{Re}\left(\alpha_0 w + \frac{\alpha_1}{2}w^2 + \frac{w^3}{3}\right),$$

plus some additive constant on each connected component of the droplet $K = \Omega^c$. Note that Q is globally admissible because $\mathrm{Re}(w^3) \lesssim |w|^4 \implies Q(w) = O(|w|^4)$. Thus by Frostman's theorem (2.1.1), for each $t > 0$ there exists a unique droplet K_t associated to Q, the complement of which is a quadratic PQD. Combining this with Lemma 2.2.2 we find that, for each choice of constants α_0 and α_1, there is an associated 1−parameter family of quadratic PQDs with $a = 2$.

When $a = 2$, the above relations reduce to $c_2 = 2\alpha_1 c^{-2} + c^{-1}\overline{c_1}$, and $c_3 = 2c^{-1}$, so that

$$\varphi(z) = cz\left(1 + \frac{\overline{c_1}}{z} + \frac{2\overline{\alpha_1} + c_1 c}{c^2 z^2} + \frac{2}{cz^3}\right)^{\frac{1}{2}}$$

Moreover, taking the aforementioned algebraic equation in c_1, along with its conjugate equation, we may eliminate $\overline{c_1}$ to obtain a quartic in c_1,

$$0 = 64(\alpha_1 + \overline{\alpha_0})^2 - 128(c^2-1)^2(\overline{\alpha_1} + \alpha_0) + 64c(c^2-1)^3 c_1 - 16c^2(\alpha_1 + \overline{\alpha_0})c_1^2 + c^4 c_1^4.$$

By numerically solving the quartic for c_1 for various values of c, α_0 and α_1, we obtain six families of quadratic PQDs with $a = 2$ which appear to exhibit distinct critical and topological behavior (Figures 4.9 and 4.10).

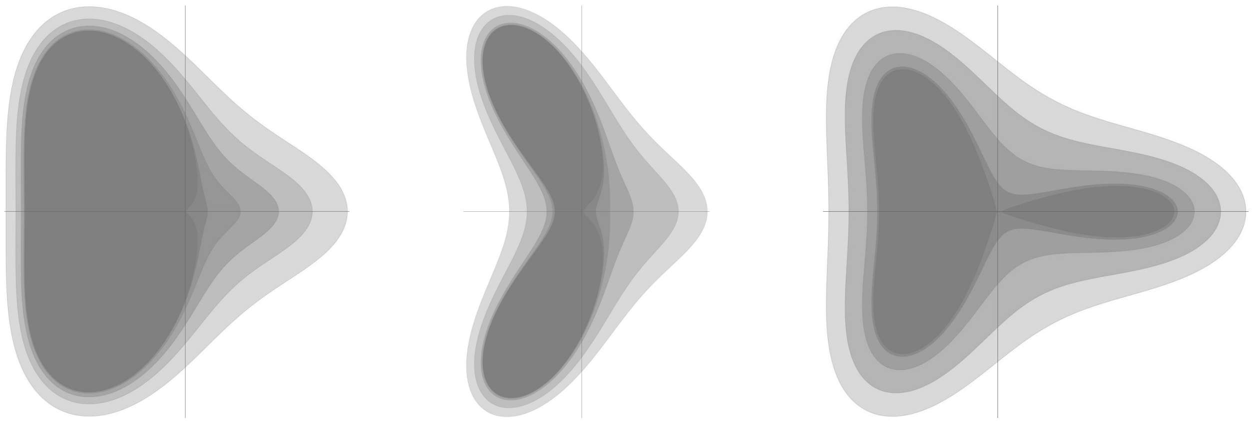

Figure 4.9: Families of PQDs (complements of the shaded regions) in $\mathrm{QD}_2(\alpha_0 + \alpha_1 w + w^2)$ for $\alpha_0 = -30$, $\alpha_1 = 0$, and $c_{\min} \approx 3.875$ (left); $\alpha_0 = -4$, $\alpha_1 = -3.04$, and $c_{\min} \approx 2.887$ (center); $\alpha_0 = -3.52$, $\alpha_1 = 1.43$, and $c_{\min} \approx 2.741$ (right).

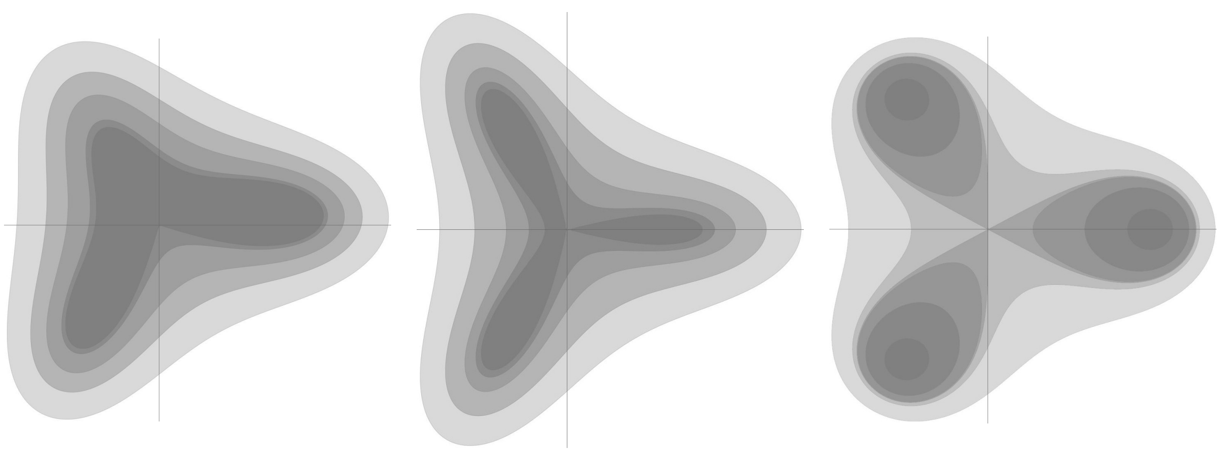

Figure 4.10: Families of PQDs (complements of the shaded regions) in $\mathrm{QD}_2(\alpha_0 + \alpha_1 w + w^2)$ for $\alpha_0 = .86$, $\alpha_1 = .96 - i$, and $c_{\min} \approx 2.6649$ (left); $\alpha_0 = -.5$, $\alpha_1 = -.17$, and $c_{\min} \approx 2.026$ (center); $\alpha_0 = 0$, $\alpha_1 = 0$, and $c_{\min} = 2$ (right).

Chapter 5

LOG-WEIGHTED QUADRATURE DOMAINS

In this section we develop the theory of *log-weighted quadrature domains* (LQDs). We begin by defining LQDs and highlighting a method of constructing LQDs from classical QDs via the exponential map (Theorem 5.1.1). We then derive the singular and non-singular coincidence equations, use them to establish the essential uniqueness of the quadrature function, and prove a generalized Schwarz-function characterization of LQDs. From this characterization we obtain boundary regularity, an electrostatic characterization, and several basic invariance properties that will be used throughout the remainder of this chapter.

5.1 Definition and Basic Properties

We consider domains $\Omega \subset \widehat{\mathbb{C}}$ such that

$$\int_{\Omega} \frac{f(w)}{|w|^2} dA(w) = \oint_{\partial\Omega} f(w)h(w)dw, \tag{5.1}$$

for all $f \in L^1_a(\Omega; \rho_0)$, with h rational. Note that when $0 \in \Omega$, admissible test functions vanish at 0; when Ω is unbounded, admissible test functions vanish at infinity. To ensure essential uniqueness of the quadrature function h, we place the additional constraint that $h \in \mathrm{Rat}_0(\Omega)$ when Ω is bounded, and $h \in \mathrm{Rat}(\Omega)$ when Ω is unbounded.

Definition 5.1.1 (Log-Weighted Quadrature Domain)**.** Let $\Omega \subset \widehat{\mathbb{C}}$ be a rectifiable domain for which $0, \infty \notin \partial\Omega$. We say that Ω is a *log-weighted quadrature domain* if there exists a rational h (satisfying the above conditions) such that Equation 5.1 is satisfied.

Throughout the remainder of the thesis we will denote by $\Omega \in \mathrm{QD}_0(h)$ that Ω is an LQD with quadrature function h, and by $\Omega \in \mathrm{QD}_0$ we denote that there exists an h for which $\Omega \in \mathrm{QD}_0(h)$. As with classical QDs, we will always require that $\Omega = \mathrm{Int}(\mathrm{Cl}(\Omega))$ because otherwise arbitrarily many QDs with the same quadrature function could be constructed via deletion of subsets of measure zero. We can generate some easy first examples of LQDs by considering the images of certain classical QDs under the exponential map. In particular,

Theorem 5.1.1. *If $\Omega \in QD(h)$ is a bounded domain on which $w \mapsto e^w$ is injective, then $e^\Omega \in QD_0(\widetilde{h})$, where $\widetilde{h}(w) = C_{e^\Omega_*}\left[\frac{h \circ \ln(w)}{w}\right]$.*

Theorem 5.1.1 also tells us that if $\{p_k\}$ are the poles of h, then $\{e^{p_k}\}$ are the poles of the quadrature function for e^Ω. It follows from the residue theorem that these poles also have the same multiplicity.

Proof of Theorem 5.1.1. Under the above hypotheses, $f \in L^1_a(e^\Omega; \rho_0) \iff f(e^w) \in \mathcal{A}(\Omega)$, so

$$\int_{e^\Omega} \frac{f(w)}{|w|^2} dA(w) = \int_\Omega f(e^w) dA(w) = \oint_{\partial\Omega} f(e^w) h(w) dw = \oint_{\partial e^\Omega} f(w) \frac{h \circ \ln(w)}{w} dw.$$

By Theorem 3.1.1, $\partial\Omega$ is piecewise C^1, which implies the same for ∂e^Ω. Hence, by Lemma 2.3.1(1),

$$\frac{h \circ \ln(w)}{w} \doteq C_{e^\Omega}\left[\frac{h \circ \ln(w)}{w}\right] + C_{e^\Omega_*}\left[\frac{h \circ \ln(w)}{w}\right],$$

where the first term is analytic in e^Ω. Moreover, as 0 is in the unbounded component of $(e^\Omega)^c$, $\frac{h \circ \ln(w)}{w} \in \mathcal{M}(e^\Omega)$ and extends continuously to ∂e^Ω, so $\widetilde{h}(w) := C_{e^\Omega_*}\left[\frac{h \circ \ln(w)}{w}\right] \in \mathrm{Rat}_0(e^\Omega)$ by Lemma 2.3.1. The C_{e^Ω} term drops out because it is analytic in e^Ω, and we obtain

$$\int_{e^\Omega} \frac{f(w)}{|w|^2} dA(w) = \oint_{\partial e^\Omega} f(w) C_{e^\Omega}\left[\frac{h \circ \ln(w)}{w}\right] dw + \oint_{\partial e^\Omega} f(w)\widetilde{h}(w) dw = \oint_{\partial e^\Omega} f(w)\widetilde{h}(w) dw,$$

and we may conclude that $e^\Omega \in \mathrm{QD}_0(\widetilde{h})$. □

For example, consider $\mathbb{D}_2(-1) \in \mathrm{QD}\left(\frac{4}{w+1}\right)$. As the exponential map is injective on $\mathbb{D}_2(-1)$, we find that $e^{\mathbb{D}_2(-1)}$ is an LQD, with quadrature function $\widetilde{h}$ having a simple pole at e^{-1}. Note that $\frac{1}{\ln(w)+1} = \frac{1}{ew-1} + G(w)$ for some $G \in \mathcal{A}(e^{\mathbb{D}_2(-1)})$. Hence, by Lemma 2.3.1(3,4),

$$\widetilde{h}(w) = C_{e_*^{\mathbb{D}_2(-1)}}\left[\frac{4}{w}\frac{1}{\ln(w)+1}\right] = C_{e_*^{\mathbb{D}_2(-1)}}\left[\frac{4}{w}\left(\frac{1}{ew-1} + G(w)\right)\right] = \frac{4}{w - e^{-1}}.$$

That is, $e^{\mathbb{D}_2(-1)} \in \mathrm{QD}_0\left(\frac{4}{w-e^{-1}}\right)$. Figure 5.1 shows $\mathbb{D}_2(-1)$ and its image under the exponential map.

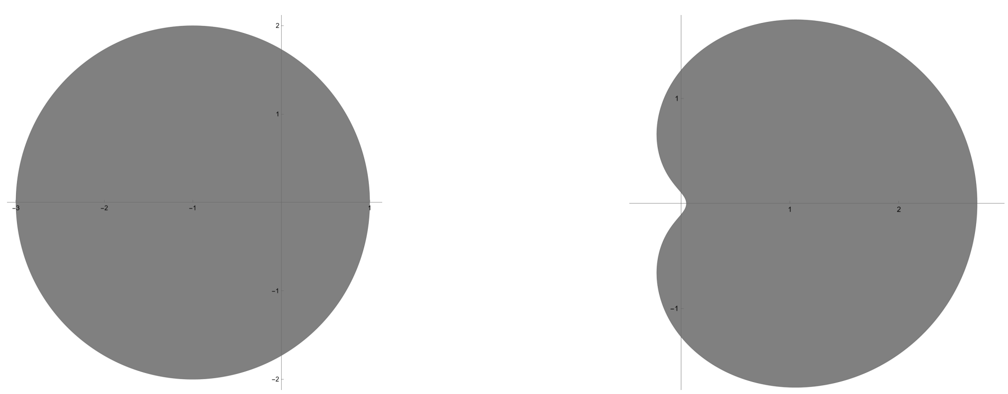

Figure 5.1: $\mathbb{D}_2(-1) \in \mathrm{QD}\left(\frac{4}{w+1}\right)$ and its image under the exponential map, $e^{\mathbb{D}_2(-1)} \in \mathrm{QD}_0\left(\frac{4}{w-e^{-1}}\right)$.

The Generalized Coincidence Equation

The primary distinction of LQDs from classical QDs is the existence of a metric singularity at $w = 0$. We refer to LQDs containing zero as *singular*. We likewise refer to LQDs not containing 0 as *non-singular*. The effect of this singularity is captured by the additional $\frac{q}{w}$ term in the *coincidence equation* for singular LQDs (Equation 5.3). Graven & Makarov [13] obtained the following generalization of the coincidence equation (Equation 3.3) for non-singular LQDs.

Theorem 5.1.2 ([13], §4)**.** *If $\Omega \in QD_0(h)$ is a domain not containing zero, then there exists a unique $G \in \mathcal{A}(\Omega)$ such that*

$$\frac{\ln|w|^2}{w} \doteq h(w) + G(w). \tag{5.2}$$

Moreover, $G(w) = C_{\rho_0}^{\Omega_*\setminus\mathbb{D}_r}(w) + \frac{\ln|r|^2}{w}$ for any $r \in (0, d(0, \partial\Omega))$.

C_ρ^U denotes the $\rho-$weighted *Cauchy transform* of $U \subseteq \widehat{\mathbb{C}}$, which is given by

$$C_\rho^U(w) = \int_U \frac{\rho(\xi)}{w - \xi} dA(\xi).$$

Obtaining a generalized coincidence equation for singular LQDs is complicated by the fact that the Cauchy kernel $k_w(\xi) = \frac{1}{\xi - w}$ is no longer an element of the test class $L_a^1(\Omega; \rho_0)$. Hence, the quadrature identity cannot be directly applied to k_w to obtain the quadrature function in terms of the weighted Cauchy transform. Instead, the modified Cauchy kernel $\frac{\xi}{\xi - w}$, vanishing at 0 is employed. We obtain the following generalization of Theorem 5.1.2 to singular LQDs.

Theorem 5.1.3. *If $\Omega \in QD_0(h)$ is a domain containing zero, then there exist unique $q \in \mathbb{C}$ and $G \in \mathcal{A}(\Omega)$ such that*

$$\frac{\ln|w|^2}{w} \doteq h(w) + \frac{q}{w} + G(w), \tag{5.3}$$

where $G = C_{\rho_0}^{\Omega_*}$.

In order to proceed with the proof of Theorems 5.1.2 and 5.1.3, we will need the following lemma which allows us to appropriately "renormalize" the weighted Cauchy transform:

Lemma 5.1.1. For all $|w| > r > 0$,

$$C_{\rho_0}^{\mathbb{C}\setminus\mathbb{D}_r}(w) = \frac{\ln|w|^2 - \ln|r|^2}{w} \tag{5.4}$$

To see this, we express the left side in Cauchy principal value and apply Green's theorem

$$\begin{aligned}
C_{\rho_0}^{\mathbb{C}\setminus\mathbb{D}_r}(w) &= \lim_{R\to\infty}\int_{r<|\xi|<R}\frac{|\xi|^{-2}}{w-\xi}dA(\xi)\\
&= \frac{\ln|w|^2}{w} + \lim_{R\to\infty}\oint_{\partial\mathbb{D}_R}\frac{\ln|\xi|^2}{\xi(w-\xi)}d\xi - \oint_{\partial\mathbb{D}_r}\frac{\ln|\xi|^2}{\xi(w-\xi)}d\xi\\
&= \frac{\ln|w|^2}{w} + \lim_{R\to\infty}\ln|R|^2\oint_{\partial\mathbb{D}_R}\frac{d\xi}{\xi(w-\xi)}d\xi - \ln|r|^2\oint_{\partial\mathbb{D}_r}\frac{d\xi}{\xi(w-\xi)}.
\end{aligned}$$

Applying the residue theorem to the latter two terms, we obtain 0 and $-\frac{\ln|r|^2}{w}$ respectively. The desired formula follows.

Proof of Theorem 5.1.2. Fix $0 \notin \Omega \in \mathrm{QD}_0(h)$ and $r < d(0,\partial\Omega)$. Note that when $w \in \Omega_*$, the quadrature identity implies

$$C_{\rho_0}^{\Omega}(w) = \int_\Omega \frac{|\xi|^{-2}}{w-\xi}dA(\xi) = \oint_{\partial\Omega}\frac{h(\xi)}{w-\xi}d\xi = h(w).$$

On the other hand if $w \in \Omega$, then $C_{\rho_0}^{\Omega_*\setminus\mathbb{D}_r}$ is analytic and extends continuously to the boundary ([4], Chapter 17). Hence, for each $w \in \partial\Omega$,

$$C_{\rho_0}^{\mathbb{C}\setminus\mathbb{D}_r}(w) = C_{\rho_0}^{\Omega_*\setminus\mathbb{D}_r}(w) + C_{\rho_0}^{\Omega}(w) = h(w) + C_{\rho_0}^{\Omega_*\setminus\mathbb{D}_r}(w).$$

Hence, by Lemma 5.1.1,

$$\frac{\ln|w|^2}{w}\doteq h(w) + \left(C_{\rho_0}^{\Omega_*\setminus\mathbb{D}_r}(w) + \frac{\ln|r|^2}{w}\right).$$

Equation 5.2 follows by setting G equal to the rightmost term. For uniqueness of G: Suppose there existed G_1 and G_2 such that Equation 5.2 holds. Subtracting these equations from each other, we immediately find that $G_1 \doteq G_2$. As G_1 and G_2 extend continuously to the boundary, it follows from the maximum modulus principle that $G_1 = G_2$ on Ω. □

The proof Theorem 5.1.3 has two steps. In the bounded case, the modified kernel $\frac{\xi}{\xi-w}$ directly yields the required decomposition. In the unbounded case, one instead uses a two-point kernel to compensate for the behavior at infinity, and the resulting constant term is again absorbed into the charge q.

Proof of Theorem 5.1.3. Fix $0 \in \Omega \in \mathrm{QD}_0(h)$ and $r < d(0,\partial\Omega)$. We first consider the case in which Ω is bounded. If $w \in \Omega_*$ then, applying the quadrature identity to $\left(\xi \mapsto \frac{\xi}{w-\xi}\right) \in L_a^1(\Omega;\rho_0)$ yields

$$\int_\Omega \frac{\xi|\xi|^{-2}}{\xi-w}dA(\xi) = \oint_{\partial\Omega}\frac{\xi h(\xi)}{\xi-w}d\xi = -wh(w).$$

On the other hand, if $w \in \Omega$ then, by Green's theorem ($\Omega \in \mathrm{QD}_0$ by assumption, so its boundary is rectifiable),

$$\int_\Omega \frac{\xi|\xi|^{-2}}{\xi - w} dA(\xi) = -\ln|w|^2 + \oint_{\partial\Omega} \frac{\ln|\xi|^2}{\xi - w} d\xi = -\ln|w|^2 + w \oint_{\partial\Omega} \frac{\ln|\xi|^2}{\xi(\xi - w)} d\xi + \oint_{\partial\Omega} \frac{\ln|\xi|^2}{\xi} d\xi.$$

Applying Green's theorem to the first integral, and splitting the second to isolate the pole at 0, we obtain

$$\int_\Omega \frac{\xi|\xi|^{-2}}{\xi - w} dA(\xi) = -\ln|w|^2 - w \int_{\Omega_*} \frac{|\xi|^{-2}}{\xi - w} d\xi + \left(\oint_{\partial(\Omega\setminus\mathbb{D}_r)} \frac{\ln|\xi|^2}{\xi} d\xi + \oint_{\partial\mathbb{D}_r} \frac{\ln|\xi|^2}{\xi} d\xi \right)$$

We recognize the first integral as a weighted Cauchy transform, and note that the second two integrals integrate to a constant depending only on Ω:

$$\int_\Omega \frac{\xi|\xi|^{-2}}{\xi - w} dA(\xi) = -\ln|w|^2 + wC^{\Omega_*}_{\rho_0}(w) + q_\Omega.$$

where $q_\Omega = \int_{\Omega\setminus\mathbb{D}_r} |\xi|^{-2} dA(\xi) + \ln|r|^2$. Hence,

$$\frac{\ln|w|^2}{w} \doteq h(w) + C^{\Omega_*}_{\rho_0}(w) + \frac{q_\Omega}{w}$$

Finally, we consider the case in which Ω is unbounded. Fix $w_0 \in \Omega_*$. If $w \in \Omega_*$ then, applying the quadrature identity to $\frac{\xi}{(w-w_0)(w-\xi)} \in L^1_a(\Omega; \rho_0)$ yields

$$\int_\Omega \frac{\xi|\xi|^{-2}}{(\xi - w_0)(\xi - w)} dA(\xi) = \oint_{\partial\Omega} \frac{\xi h(\xi)}{(\xi - w_0)(\xi - w)} d\xi = -\frac{wh(w) - w_0 h(w_0)}{w - w_0}.$$

On the other hand, if $w \in \Omega$ then, by Green's theorem,

$$\begin{aligned} \int_\Omega \frac{\xi|\xi|^{-2}}{(\xi - w_0)(\xi - w)} dA(\xi) &= -\frac{\ln|w|^2}{w - w_0} + \oint_{\partial\Omega} \frac{\ln|\xi|^2}{(\xi - w_0)(\xi - w)} d\xi \\ &= \frac{-1}{w - w_0} \left(\ln|w|^2 - w \oint_{\partial\Omega} \frac{\ln|\xi|^2}{\xi(\xi - w)} d\xi + w_0 \oint_{\partial\Omega} \frac{\ln|\xi|^2}{\xi(\xi - w_0)} d\xi \right) \end{aligned}$$

Applying Green's theorem to the first integral, and splitting the second to isolate the pole at 0, we obtain

$$\frac{-1}{w - w_0} \left(\ln|w|^2 + w \int_{\Omega_*} \frac{|\xi|^{-2}}{\xi - w} dA(\xi) + w_0 \left(\oint_{\partial(\Omega\setminus\mathbb{D}_r)} \frac{\ln|\xi|^2 d\xi}{\xi(\xi - w_0)} + \oint_{\partial\mathbb{D}_r} \frac{\ln|r|^2 d\xi}{\xi(\xi - w_0)} \right) \right)$$

We recognize the first integral as a weighted Cauchy transform, the second integral is also recognized as a weighted Cauchy transform after applying Green's theorem, and the third integral may be calculated directly via the residue theorem.

$$\begin{aligned} \int_\Omega \frac{\xi|\xi|^{-2}}{(\xi - w_0)(\xi - w)} dA(\xi) &= \frac{-1}{w - w_0} \left(\ln|w|^2 - wC^{\Omega_*}_{\rho_0}(w) + w_0 \left(\int_{\Omega\setminus\mathbb{D}_r} \frac{|\xi|^{-2}}{\xi - w_0} dA(\xi) - \frac{\ln|r|^2}{w_0} \right) \right) \\ &= \frac{-1}{w - w_0} \left(\ln|w|^2 - wC^{\Omega_*}_{\rho_0}(w) - w_0 C^{\Omega\setminus\mathbb{D}_r}_{\rho_0}(w_0) + \ln|r|^2 \right). \end{aligned}$$

Hence for each $w \in \partial\Omega$,

$$\frac{\ln|w|^2}{w} \doteq h(w) + C^{\Omega_*}_{\rho_0}(w) + \frac{q_{\Omega,w_0}}{w},$$

where $q_{\Omega,w_0} = w_0 C^{\Omega\setminus\mathbb{D}_r}_{\rho_0}(w_0) - w_0 h(w_0) + \ln|r|^2$. For the uniqueness of G and q, suppose there existed two pairs G_1, q_1 and G_2, q_2 for which Equation 5.3 holds. Subtracting these from each other, and multiplying through by w, we find that $wG_1(w) + q_1 \doteq wG_2(w) + q_2$. Both sides of the equation are analytic in Ω and extend continuously to the boundary, it follows from the identity theorem that $wG_1(w) + q_1 = wG_2(w) + q_2$ on Ω. Plugging in $w = 0$, we find that $q_1 = q_2$, from which it follows that $G_1 = G_2$ on Ω. □

With the generalized coincidence equation established, our next goal is to characterize the conditions under which the quadrature function h associated to a given LQD is unique. In particular, we show that h is unique outright when $0 \notin \Omega$, and unique modulo an additional point-charge term q/w when $0 \in \Omega$. Next, we recover an analogue of the Schwarz function for LQDs (Proposition 3.1.1): we introduce a *generalized Schwarz function* S_0 whose boundary values satisfy $S_0(w) \doteq \frac{\ln|w|^2}{w}$, and show that Ω is an LQD if and only if such an S_0 exists. We then record three further consequences that will be used repeatedly later: a Sakai-type boundary regularity theorem (only finitely many singular boundary points, each a cusp or double point), an electrostatic characterization of LQDs, and basic invariance properties of the class under scaling, inversion, and power maps.

Essential Uniqueness of the Quadrature Function

A direct consequence of Theorems 5.1.2 and 5.1.3 is the essential uniqueness of the quadrature function associated to a given LQD. For non-singular LQDs, we obtain:

Corollary 5.1.1. If $\Omega \in \mathrm{QD}_0(h)$ is non-singular, then h is unique.

Proof. Suppose $0 \notin \Omega$ is an LQD with two quadrature functions h_1 and h_2 (that is, $\Omega \in \mathrm{QD}_0(h_1) \cap \mathrm{QD}_0(h_2)$). Then, h_1 and h_2 each satisfy Equation 5.2 for the same G, as G is independent of h. Subtracting these equations, we obtain $h_1 \doteq h_2$. It follows from the identity theorem that $h_1 = h_2$. □

On the other hand, if $0 \in \Omega$, then we only obtain uniqueness up to a one-parameter family of quadrature functions.

Corollary 5.1.2. If $\Omega \in \mathrm{QD}_0(h)$ is a domain containing zero, then h is unique modulo the addition of a point charge $\frac{q}{w}$ at the origin. In particular, $\left\{\widetilde{h} : \Omega \in \mathrm{QD}_0(\widetilde{h})\right\} = \left\{h(w) + \frac{q}{w} : q \in \mathbb{C}\right\}$.

Proof. Suppose $0 \in \Omega$ is an LQD with two quadrature functions h_1 and h_2 (that is, $\Omega \in \mathrm{QD}_0(h_1) \cap \mathrm{QD}_0(h_2)$). Then, h_1 and h_2 each satisfy Equation 5.3. Noting that G is independent of h (it depends only on the domain), we obtain $\frac{\ln|w|^2}{w} \doteq h_j(w) + \frac{q_j}{w} + G(w)$ ($q_j \in \mathbb{C}$, $j \in \{1, 2\}$). Subtracting these equations and applying the identity theorem yields $h_1(w) - h_2(w) = \frac{q_2 - q_1}{w}$ for all $w \in \mathbb{C}$. In particular, there exists $q \in \mathbb{C}$ such that $h_2(w) = h_1(w) + \frac{q}{w}$.

On the other hand, if $\Omega \in \mathrm{QD}_0(h)$, then $\Omega \in \mathrm{QD}_0\left(h(w) + \frac{q}{w}\right)$ for all $q \in \mathbb{C}$ by inspection of the quadrature identity, using the fact that $f \in L^1_a(\Omega; \rho_0) \Rightarrow f(0) = 0$. □

Considering Theorem 5.1.3 together with Corollary 5.1.2, we arrive at an apparent paradox: The former asserts that q (in Equation 5.3) is uniquely associated to $\Omega \in \mathrm{QD}_0(h)$, whereas the latter asserts that $h(w) + \frac{q}{w}$ is also a quadrature function for Ω for any $q \in \mathbb{C}$. What is happening here? The root of the problem is that q (in Equation 5.3) depends on both Ω and h. Hence we must instead consider the sum $h(w) + \frac{q}{w}$ which is, indeed uniquely associated to Ω (this follows from essentially the same argument as in the proof of Corollary 5.1.2). Now consider the decomposition

$$h(w) + \frac{q}{w} = \left(h(w) - \frac{\operatorname{Res}_0 h}{w} \right) + \frac{q + \operatorname{Res}_0 h}{w},$$

where the first term is the part which is residue-free at 0, and the second captures the residue at 0. It follows that the second term is, in fact, independent of h. This motivates the introduction of the following definition.

Definition 5.1.2. Fix $q \in \mathbb{C}$ and h rational such that $\operatorname{Res}_0 h = 0$. We denote by $\Omega \in \mathrm{QD}_0(h; q)$ that $\Omega \in \mathrm{QD}_0(h)$, and Equation 5.3 is satisfied with this value of q.

It is immediate that if $\Omega \in \mathrm{QD}_0(h; q)$, then the pair (h, q) is uniquely associated to Ω. From the perspective of potential theory, q can be interpreted as the charge at 0.

The Generalized Schwarz Function and Boundary Regularity

Recall that quadrature domains may be equivalently characterized as those domains which admit a Schwarz function (Proposition 3.1.1). The coincidence equations suggest that the correct analogue of the classical Schwarz function should have boundary values $\frac{\ln|w|^2}{w}$. We now show that this is indeed the right notion. In particular, we show that a domain Ω is an LQD if and only if it admits a *generalized Schwarz function*: A function $S_0 \in \mathcal{M}(\Omega)$ extending continuously to $\partial\Omega$ such that $S_0(w) \doteq \frac{\ln|w|^2}{w}$.

It is important to note that S_0 *always has finitely many poles* - a fact which is clear when Ω is bounded. When Ω is unbounded, this follows from the fact that S_0 is meromorphic at ∞, which means that the singularity at infinity (if any) is isolated.

Theorem 5.1.4. *If Ω is a domain with $0, \infty \notin \partial\Omega$, then $\Omega \in QD_0$ if and only if it admits a generalized Schwarz function S_0.*

In this case, the generalized Schwarz function S_0 is given by the RHS of Equation 5.2 when $0 \notin \Omega$ and RHS of Equation 5.3 when $0 \in \Omega$. In order to complete the proof of Theorem 5.1.4, we will need the following regularity result:

Lemma 5.1.2. If Ω admits a generalized Schwarz function, then $\partial\Omega$ has finitely many singular points, and each is a cusp or a double point.

Proof. Let S_0 be a generalized Schwarz function for Ω. Define

$$S(w) := \frac{e^{wS_0(w)}}{w}.$$

For $w \in \partial\Omega$, the boundary condition for S_0 gives

$$S(w) \doteq \frac{e^{\ln|w|^2}}{w} = \frac{|w|^2}{w} = \overline{w}.$$

Since $S_0 \in \mathcal{M}(\Omega)$ and is continuous up to $\partial\Omega$, its poles are finite in number and bounded away from $\partial\Omega$. Choose $\epsilon > 0$ sufficiently small that S_0 is holomorphic in

$$N_\epsilon = \{w \in \Omega : d(w, \partial\Omega) < \epsilon\},$$

and $d(0, \partial\Omega) > \epsilon$. Then S is holomorphic on $N_\epsilon \cap \mathrm{Cl}(\Omega)$ with boundary values $S(w) \doteq \overline{w}$. Thus S is a local Schwarz function at every boundary point of Ω. Sakai's regularity theorem (3.1.1) then implies that $\partial\Omega$ has finitely many singular points, each a cusp or a double point ($\partial\Omega$ has no degenerate points on account of the assumption that $\Omega = \mathrm{Int}(\mathrm{Cl}(\Omega))$). The finiteness of the singular set follows exactly as in the classical quadrature-domain case (§3.2 of [20]). □

With this regularity result in hand, we are now prepared to prove Theorem 5.1.4.

Proof of Theorem 5.1.4. Suppose $\Omega \in \mathrm{QD}_0(h)$. Then by Theorems 5.1.2 and 5.1.3, Ω satisfies Equation 5.3 (with $q = 0$ when $0 \notin \Omega$). Set S_0 equal to the RHS of the equation: $S_0(w) := h(w) + qw^{-1} + G(w)$. As $S_0 \in \mathcal{M}(\Omega)$ and extends continuously to the boundary ($0 \notin \partial\Omega$ by assumption), we find that $S_0(w) \doteq \frac{\ln|w|^2}{w}$ is a generalized Schwarz function for Ω.

On the other hand, suppose Ω admits a generalized Schwarz function S_0. By Lemma 5.1.2, we know that $\partial\Omega$ is piecewise C^1, so Green's theorem applies. Hence, for each $f \in L^1_a(\Omega; \rho_0)$,

$$\int_\Omega \frac{f(w)}{|w|^2} dA(w) = \oint_{\partial\Omega} f(w) \frac{\ln|w|^2}{w} dw = \oint_{\partial\Omega} f(w) S_0(w) dw.$$

Since S_0 has finitely many poles in Ω, let h be the rational function obtained by summing the principal parts of S_0 at those poles, normalized so that $h \in \mathrm{Rat}_0(\Omega)$ in the bounded case and $h \in \mathrm{Rat}(\Omega)$ in the unbounded case. Then

$$G = S_0 - h$$

is holomorphic in Ω and extends continuously to $\partial\Omega$. Hence,

$$\int_{\partial\Omega} f(w)G(w)dw = 0$$

by Cauchy's theorem. It follows that

$$\int_{\Omega} \frac{f(w)}{|w|^2} dA(w) = \oint_{\partial\Omega} f(w)h(w)dw,$$

so $\Omega \in \mathrm{QD}_0$. □

An immediate consequence of Theorem 5.1.4 and Lemma 5.1.2 is the following boundary regularity result for LQDs:

Theorem 5.1.5. *If $\Omega \in QD_0$ then $\partial\Omega$ has finitely many singular points, and each is a cusp or a double point. In particular, $\partial\Omega$ is piecewise C^1.*

Proof. By Theorem 5.1.4, Ω has a generalized Schwarz function. The conclusion follows by Lemma 5.1.2. □

LQDs: An Electrostatic Perspective

LQDs have a natural physical interpretation from the perspective of electrostatics. In particular, *a domain Ω is an LQD if and only if there is a finite configuration of point charges (and multipoles) on the interior of Ω whose electrostatic field in Ω^c is equal to that of a charge density ρ_0 on Ω.* More formally,

Theorem 5.1.6. *Fix a domain Ω such that $0, \infty \notin \partial\Omega$, and $r \in (0, d(0, \partial\Omega))$. Then,*

1. *If $0 \notin \Omega$ then $\Omega \in QD_0(h)$ if and only if $C^{\Omega}_{\rho_0}(w) = h(w)$ on Ω_*;*
2. *If $0 \in \Omega$ then $\Omega \in QD_0(h; q)$ if and only if $C^{\Omega\setminus\mathbb{D}_r}_{\rho_0}(w) + \frac{\ln|r|^2}{w} = h(w) + \frac{q}{w}$ on Ω_*.*

See Appendix A.1 for the proof.

For example, when $0 \notin \Omega$, $\overline{C^{\Omega}_{\rho_0}}$ is precisely the electrostatic field due to the charge density ρ_0 on Ω, and $\overline{h}$ is the electrostatic field due to a collection of point charges (and multipoles) at the poles of h. This phenomenon is illustrated in Figure 5.2.

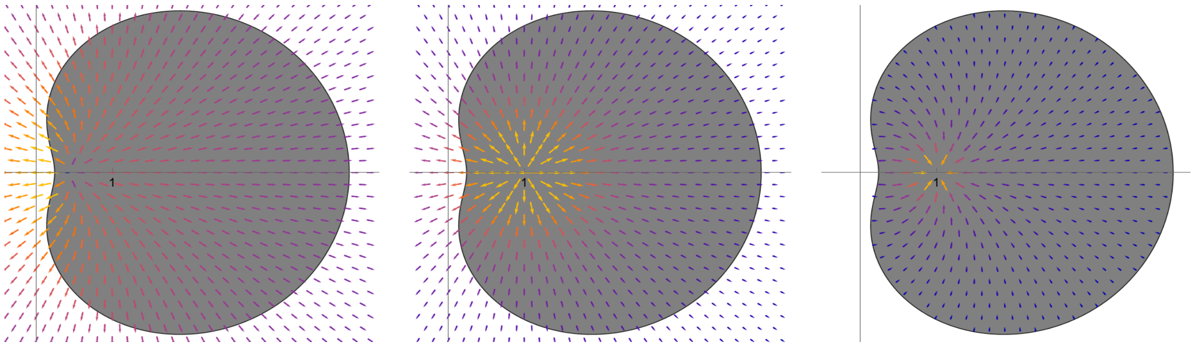

Figure 5.2: The electrostatic field for $\Omega \in \mathrm{QD}_0\,(h)$ $(h(w) = \frac{2}{w-1})$ for $\overline{C^{\Omega}_{\rho_0}}$ (left), $\overline{h}$ (center), and $\overline{C^{\Omega}_{\rho_0}} - \overline{h}$ (right).

Invariance Properties

The class of log-weighted quadrature domains has a number of invariance properties. In particular, if $a \in \mathbb{C} \setminus \{0\}$, and $k \in \mathbb{Z}_+$, then

1. Scale invariance (§5.1): $\Omega \in \mathrm{QD}_0 \iff a^{-1}\Omega \in \mathrm{QD}_0$,
2. Inversion invariance (§5.1): $\Omega \in \mathrm{QD}_0 \iff \Omega^{-1} \in \mathrm{QD}_0$,[1]
3. Power invariance (§5.1): $\Omega \in \mathrm{QD}_0 \iff \Omega^{\frac{1}{k}} \in \mathrm{QD}_0$, (assuming $\Omega^{\frac{1}{k}}$ is connected),

where $\Omega^{\frac{1}{k}} := \{w \in \widehat{\mathbb{C}} : w^k \in \Omega\}$ is the full kth root preimage of Ω. The specific relationships between the quadrature functions of these domains and their transformed counterparts are given in the §5.1-5.1. Some of these properties are genuinely new while others are, in some sense, "inherited" from invariance properties of classical quadrature domains via Theorem 5.1.1. For example, the scale-invariance property of QD_0 can be interpreted as a consequence of the translation invariance of QD on account of the fact that the exponential map takes addition to multiplication. Of course, LQDs are not generally translation invariant, as this is not a symmetry of the underlying weight $\rho_0(w) = |w|^{-2}$.

We will begin by covering scale invariance and its implications for "traveling wave" families of LQDs.

[1]Where $\Omega^{-1} = \{w^{-1} \in \widehat{\mathbb{C}} : w \in \Omega\}$.

Scaling Law and Traveling Waves

Fix $\alpha \in \mathbb{C} \setminus \{0\}$ and $\Omega \in \mathrm{QD}_0(h)$, and note that $f \in L^1_a(a^{-1}\Omega;\rho_0) \implies f(a^{-1}w) \in L^1_a(\Omega;\rho_0)$. Applying the quadrature identity, we obtain

$$\int_{a^{-1}\Omega} \frac{f(w)}{|w|^2} dA(w) = \int_{\Omega} \frac{f(a^{-1}w)}{|a^{-1}w|^2} |a|^{-2} dA(w) = \oint_{\partial\Omega} f(a^{-1}w)h(w)dw = \oint_{\partial a^{-1}\Omega} f(w)h(aw)adw.$$

Hence, $a^{-1}\Omega \in \mathrm{QD}_0(h(aw)a)$. Carrying out the same argument with a^{-1}, we conclude that

Theorem 5.1.7. *For each $a \in \mathbb{C} \setminus \{0\}$ and domain Ω, $\Omega \in QD_0(h)$ if and only if $a^{-1}\Omega \in QD_0(h(aw)a)$.*

We say that a family $\{\Omega_t\}_t \subseteq \mathrm{QD}_0(h)$ is a traveling wave if there exists $t_0 > 0$ and a smooth increasing function $\alpha(t)$ such that $\alpha(\infty) = \infty$ and for each $t > t_0$, $\Omega_t = \alpha(t)\Omega_{t_0}$.

Corollary 5.1.3. If $\{\Omega_t\}_{0<t<\infty} \subseteq \mathrm{QD}_0(h)$ is a traveling wave, then it consists of either disks or exterior disks centered at the origin.

Proof. By Theorem 5.1.7, $\Omega_{t_0} \in \mathrm{QD}_0(h) \cap \mathrm{QD}_0(h(\alpha(t)w)\alpha(t))$. Then by Corollaries 5.1.1 and 5.1.2, either

1. $h(w) = h(\alpha(t)w)\alpha(t)$, when $0 \notin \Omega$,
2. $h(w) = h(\alpha(t)w)\alpha(t) + \frac{q}{w}$ for some $q \in \mathbb{C}$, when $0 \in \Omega$.

Applying the assumption that h is rational, and expanding each functional equation in a Laurent series about $w = 0$, we find that $h(w) = \frac{c}{w}$ for some $c \in \mathbb{C}$.

In the first case, we find that $c = 0$ because $h \in \mathcal{A}(\Omega_*)$ and $0 \in \Omega_*$. Hence, $\Omega_t \in \mathrm{QD}_0(0)$ for all $t > t_0$. By Theorem 5.3.1, each Ω_t must be an exterior disk centered at the origin. Similarly, in the second case, we find that $\Omega_t \in \mathrm{QD}_0\left(\frac{c}{w}\right)$. By Corollary 5.1.2 we conclude that $\Omega_t \in \mathrm{QD}_0(0)$. Again by Theorem 5.3.1, each Ω_t must be a disk centered at the origin. $\square$

Invariance Under Inversion

Another interesting property of the class of LQDs is their invariance under inversion. This is essentially a consequence of the fact that $(w^{-1})^* \frac{dA(w)}{|w|^2} = \frac{|-w^{-2}|^2 dA(w)}{|w^{-1}|^2} = \frac{dA(w)}{|w|^2}$, where $(w^{-1})^*$ is the pullback by $w \mapsto w^{-1}$. The following theorem is a direct generalization of the Graven & Makarov's [13] result for non-singular LQDS.

Theorem 5.1.8. *If $\Omega \subseteq \widehat{\mathbb{C}}$ then $\Omega \in QD_0(h)$ if and only if $\Omega^{-1} \in QD_0(-h(w^{-1})w^{-2})$.*

Proof of Theorem 5.1.8. If $\Omega \in \mathrm{QD}_0(h)$ then $f \in L^1_a(\Omega^{-1};\rho_0) \implies f(w^{-1}) \in L^1_a(\Omega;\rho_0)$. Taking the following integral, inverting, applying the quadrature identity, and inverting back, we obtain

$$\begin{aligned}\int_{\Omega^{-1}} \frac{f(w)}{|w|^2} dA(w) &= \int_{\Omega} \frac{f(w^{-1})}{|w|^2} dA(w) \\ &= \oint_{\partial\Omega} f(w^{-1})h(w)dw \\ &= \oint_{\partial\Omega^{-1}} f(w)(-h(w^{-1})w^{-2})dw\end{aligned}$$

so $\Omega^{-1} \in \mathrm{QD}_0\left(-h(w^{-1})w^{-2}\right)$. This implies the backwards direction as well because $\widetilde{h}(w) = -h(w^{-1})w^{-2} \implies -\widetilde{h}(w^{-1})w^{-2} = h(w)$. ◻

When $\Omega \in \mathrm{QD}_0(h;q)$ is singular and unbounded (in which case $\Omega^{-1} \in \mathrm{QD}_0(\widetilde{h},\widetilde{q})$ is likewise singular and unbounded), we obtain a direct relation between q and $\widetilde{q}$ in terms of the ρ_0-weighted area of Ω_*, $A_{\rho_0}(\Omega_*) = \int_{\Omega_*} |\xi|^{-2} dA(\xi)$.

Corollary 5.1.4. If $\Omega \subseteq \widehat{\mathbb{C}}$ contains both 0 and ∞, then $\Omega \in \mathrm{QD}_0(h;q)$ if and only if

$$\Omega^{-1} \in \mathrm{QD}_0(-h(w^{-1})w^{-2}; -q - A_{\rho_0}(\Omega_*)).$$

Proof. That $q \mapsto -q$, follows by substituting $w \mapsto w^{-1}$ in Equation 5.3, and multiplying through by $-w^{-2}$ to obtain

$$\frac{\ln|w|^2}{w} \doteq -h(w^{-1})w^{-2} - \frac{q}{w} - \frac{G(w^{-1})}{w^2}.$$

Note that $\frac{G(w^{-1})}{w^2} \in \mathcal{A}(\Omega^{-1})$, unless $\infty \in \Omega$. In this case, $0 \in \Omega^{-1}$, and $G(\infty) = 0$, so

$$-\frac{G(w^{-1})}{w^2} = \frac{\operatorname{Res}_\infty G}{w} - \frac{G(w^{-1}) + w\operatorname{Res}_\infty G}{w^2} = \frac{\operatorname{Res}_\infty G}{w} + \widetilde{G}(w),$$

where $\widetilde{G} \in \mathcal{A}(\Omega^{-1})$. Finally, note that

$$G(w) = \int_{\Omega_*} \frac{|\xi|^{-2}dA(\xi)}{w-\xi} = \frac{1}{w}\int_{\Omega_*} |\xi|^{-2}dA(\xi) + O(w^{-2}),$$

so $\operatorname{Res}_\infty G = -A_{\rho_0}(\Omega_*)$. ◻

An important consequence of inversion invariance is that it gives a bijective correspondence between bounded singular LQDs and unbounded non-singular LQDs. Hence, we can study one family by considering corresponding domains in the other.

The Power Map

Considering that $\ln|w^k| = k\ln|w|$, another natural transformation to investigate is the power map $w \mapsto w^k$. For a domain $\Omega \subseteq \widehat{\mathbb{C}}$, we therefore introduce its full kth root preimage, $\Omega^{\frac{1}{k}} := \{w \in \widehat{\mathbb{C}} : w^k \in \Omega\}$, which is $k-$fold rotationally symmetric. The theorem below shows that $\Omega \in \mathrm{QD}_0$ if and only if $\Omega^{\frac{1}{k}}$ is a disjoint union of LQDs (if $\Omega^{\frac{1}{k}}$ is connected, then it is a disjoint union of one LQD). The key point is that the weight $\rho_0(w) = |w|^{-2}$ interacts especially well with the change of variables induced by $w \mapsto w^k$, so the log-weighted quadrature identity is preserved. In particular, this result gives a systematic way to generate symmetric examples and, conversely, to represent symmetric LQDs via simpler ones.

Theorem 5.1.9. *Fix a domain Ω, and $k \in \mathbb{Z}_+$. Then $\Omega \in QD_0 \iff \Omega^{\frac{1}{k}} = \bigsqcup_j \Omega_j$, where $\Omega_j \in QD_0$. In this case, there exist rational functions h and $\{h_j\}_j$ such that $\Omega \in QD_0(h)$, $\Omega_j \in QD_0(h_j)$, and $\sum_j h_j = \frac{1}{k}w^{k-1}h(w^k)$.*

When $\Omega^{\frac{1}{k}}$ is connected, Theorem 5.1.9 reduces to the statement $\Omega \in QD_0(h) \iff \Omega^{\frac{1}{k}} \in QD_0\left(\frac{1}{k}w^{k-1}h(w^k)\right)$. Before proceeding with the proof, we will need the following lemma (proof in Appendix A.4) which asserts a that $k-$fold rotational symmetry of an LQD implies a certain symmetry of its quadrature function.

Lemma 5.1.3. If $\Omega = \bigsqcup_j \Omega_j$, where $\Omega_j \in \mathrm{QD}_0(h_j)$, is a $k-$fold rotationally symmetric disjoint finite union of LQDs, then there exists a rational function g such that $\sum_j h_j(w) =: h(w) = w^{k-1}g(w^k)$.

Proof of Theorem 5.1.9. For the forward direction, we suppose that $\Omega \in \mathrm{QD}_0(h)$. By Theorem 5.1.4, Ω admits a generalized Schwarz function $S_0(w) \doteq \frac{\ln|w|^2}{w}$. We then consider $w \in \partial\Omega^{\frac{1}{k}}$ ($\implies w^k \in \partial\Omega$), so

$$\widetilde{S_0}(w) := \frac{1}{k}w^{k-1}S_0(w^k) = \frac{1}{k}w^{k-1}\frac{\ln|w^k|^2}{w^k} = \frac{\ln|w|^2}{w}.$$

Fix a connected component $\Omega_j \subseteq \Omega^{\frac{1}{k}}$ and note that at each $w \in \Omega^{\frac{1}{k}} \supseteq \Omega_j$ ($\implies w^k \in \Omega$), $\widetilde{S_0}$ is meromorphic in Ω_j as the product and composition of a meromorphic function with a polynomial. It also inherits continuity up to the boundary from S_0. Hence, $\widetilde{S_0}$ is a generalized Schwarz function for each connected component Ω_j of $\Omega^{\frac{1}{k}}$, and we conclude that $\Omega^{\frac{1}{k}}$ is a disjoint union of LQDs.

For the reverse direction, suppose that $\Omega^{\frac{1}{k}} = \bigsqcup_j \Omega_j$, where $\Omega_j \in \mathrm{QD}_0(h_j)$. As $\Omega^{\frac{1}{k}}$ is $k-$fold symmetric about the origin, Lemma 5.1.3 implies that there exists a rational function g such that

$$\sum_j h_j(w) =: h(w) = \frac{1}{k}w^{k-1}g(w^k).$$

If $w^k \in \Omega$, that means that there exists $z \in \Omega^{\frac{1}{k}}$ such that $w^k = z^k$. In particular, $e^{2\pi i \frac{j}{k}} w \in \Omega^{\frac{1}{k}}$ for some $j \in \mathbb{Z}$. As $\Omega^{\frac{1}{k}}$ is $k-$fold rotationally symmetric, this means that $e^{2\pi i \frac{j}{k}} w \in \Omega^{\frac{1}{k}}$ for every $j \in \mathbb{Z}$. In particular, $\Omega^{\frac{1}{k}}$ is a $k-$to-1 cover of Ω under the map $w \mapsto w^k$ (and hence on the boundary as well). Fix $f \in L^1_a(\Omega; \rho_0)$, so that $f(w^k) \in L^1_a(\Omega^{\frac{1}{k}}; \rho_0)$. Then,

$$\begin{aligned}
\int_\Omega \frac{f(w)}{|w|^2} dA(w) &= \frac{1}{k} \int_{\Omega^{\frac{1}{k}}} \frac{f(w^k)}{|w^k|^2} k^2 |w|^{2(k-1)} dA(w) \\
&= k \sum_j \int_{\Omega_j} \frac{f(w^k)}{|w|^2} dA(w) \\
&= k \sum_j \oint_{\partial\Omega_j} f(w^k) h_j(w) dw
\end{aligned}$$

As each h_j is residue-free in Ω_j^c, we may deform each contour $\partial\Omega_j$ to the contour $\partial\Omega^{\frac{1}{k}}$ without changing the value of the integral. Doing so, we obtain

$$\int_\Omega \frac{f(w)}{|w|^2} dA(w) = k \oint_{\partial\Omega^{\frac{1}{k}}} f(w^k) \sum_j h_j(w) dw = \oint_{\partial\Omega^{\frac{1}{k}}} f(w^k) w^{k-1} g(w^k) dw = \oint_{\partial\Omega} f(w) g(w) dw.$$

That is, $\Omega \in \mathrm{QD}_0(g)$. The desired conclusion follows. □

5.2 LQDs as Limits of PQDs

Theorems 5.1.2 and 5.1.3 give an analogue of the Schwarz function equation (3.3) à la that for PQDs (4.3). There are other results for PQDs which extend nicely to this "limiting" situation. This understanding of LQDs as the limit of a sequence of PQDs as $a \longrightarrow 0^+$ is formalized in the following theorem:

Theorem 5.2.1. *Let $\{\Omega_a\}_{0<a<a_*}$ be a family of domains and $h \in \mathrm{Rat}_0\left(Int\left(\cap_a \Omega_a\right)\right)$ such that $\Omega_a \in QD_a(h)$ and the lengths of $\{\partial\Omega_a\}_a$ are uniformly bounded. If $\Omega_a \xrightarrow{a \to 0^+} \Omega_0$ in the Hausdorff metric and $0 \notin Cl(\Omega_0)$, then $\Omega_0 \in QD_0(h)$.*

Proof of Theorem 5.2.1. As $\Omega_a \xrightarrow{a \to 0^+} \Omega_0$ in the Hausdorff metric, for each $\epsilon > 0$ there exists $0 < a_\epsilon < \min\{a_*, \epsilon\}$ for which $d_H(\Omega_a, \Omega_0) < \epsilon$ for all $0 < a \le a_\epsilon$. Take ϵ_0 sufficiently small that 0 is bounded away from the Ω_a by distance $\delta = \delta(\epsilon_0) > 0$ for all $0 < a < a_\epsilon$, $0 < \epsilon \le \epsilon_0$ then, by the uniform boundedness of the boundary lengths, we have that $\mu(\Omega_a \setminus \Omega_0), \mu(\Omega_0 \setminus \Omega_a) < \frac{L\pi\epsilon^2}{\delta}$ for all $0 < a < a_\epsilon$. If $U_{\epsilon_0} = \bigcup_{0<a\le a_{\epsilon_0}} \Omega_a$ is unbounded and $f \in \mathcal{A}_0\left(U_{\epsilon_0}\right)$ is given by $f(w) = f_1 w^{-1} + f_2 w^{-2} + \ldots$ then

$$\begin{aligned}
&\left| \int_{\Omega_0} \frac{f(w)}{|w|^2} dA(w) - \int_{\Omega_a} \frac{f(w)}{|w|^2} |w|^{2a} dA(w) \right| = \left| \int_{\Omega_0 \cup \Omega_a} \frac{f(w)}{|w|^2} \left(\mathbb{1}_{\Omega_0} - |w|^{2a} \mathbb{1}_{\Omega_a} \right) dA(w) \right| \\
&= \left| \int_{\Omega_0 \cup \Omega_a} \frac{\frac{f(w)}{w}}{\overline{w}} \left(\mathbb{1}_{\Omega_0 \setminus \Omega_a} - \mathbb{1}_{\Omega_a \setminus \Omega_0} + (1 - |w|^{2a}) \mathbb{1}_{\Omega_a} \right) dA(w) \right|.
\end{aligned}$$

By the triangle inequality, this is bounded by

$$\begin{aligned}
&\left|\int_{\Omega_0\setminus\Omega_a} \frac{f(w)}{|w|^2}dA(w)\right| + \left|\int_{\Omega_a\setminus\Omega_0} \frac{f(w)}{|w|^2}dA(w)\right| + \left|\int_{\Omega_a} f(w)\frac{|w|^{2a}-1}{|w|^2}dA(w)\right| \\
&\le 2\frac{L\pi\epsilon^2}{\delta}\sup_{U_{\epsilon_0}}|f| + \left|\int_{\Omega_a}\frac{1}{|w|^2}\sum_{k=1}^{\infty}\frac{f_k}{w^k}\sum_{n=1}^{\infty}\frac{2^n a^n}{n!}\log^n|w|dA(w)\right| \\
&\le M(\epsilon_0)\epsilon^2 + \sum_{n,k=1}^{\infty}\frac{2^n a^n}{n!}|f_k|\int_{\Omega_a}\frac{|\log^n|w||}{|w|^{k+2}}dA(w) \xrightarrow{\epsilon\to 0} 0,
\end{aligned}$$

where the second integral goes to 0 with ϵ because $a_\epsilon \xrightarrow{\epsilon\to 0} 0$ and $0 \notin \mathrm{Cl}(\Omega_a)$. The argument in the bounded case is similar. Thus, for each $\epsilon_0 > 0$ sufficiently small with $0 < \epsilon \le \epsilon_0$, $0 < a < a_\epsilon < \min\{a_*, \epsilon\}$ and $f \in \mathcal{A}\left(U_{\epsilon_0}\right)$ ($\mathcal{A}_0$ if U_{ϵ_0} is unbounded),

$$\int_{\Omega_0}\frac{f(w)}{|w|^2}dA(w) = \lim_{\epsilon\to 0}\int_{\Omega_a} f(w)|w|^{2(a-1)}dA(w) = \lim_{\epsilon\to 0}\oint_{\partial\Omega_a} f(w)h(w)dw = \oint_{\partial\Omega_0} f(w)h(w)dw.$$

Now if w is any point in $(U_{\epsilon_0})_*$, $\frac{1}{\xi-w} \in \mathcal{A}\left(U_{\epsilon_0}\right)$ ($\mathcal{A}_0$ if U_{ϵ_0} is unbounded), so

$$\begin{aligned}
0 &= \int_{\Omega_0}\frac{dA(\xi)}{|\xi|^2(\xi-w)} - \int_{\Omega_0}\frac{dA(\xi)}{|\xi|^2(\xi-w)} = \oint_{\partial\Omega_0}\frac{\ln|\xi|^2 d\xi}{\xi(\xi-w)} - \oint_{\partial\Omega_0}\frac{h(\xi)d\xi}{\xi-w} = -\oint_{\partial\Omega_0}\frac{h(\xi)-\frac{\ln|\xi|^2}{\xi}}{\xi-w}d\xi \\
&= C_{(\Omega_0)_*}\left[h(w) - \frac{\ln|w|^2}{w}\right]
\end{aligned}$$

But for each $w \in \Omega_0$, there exists $\epsilon_0 > 0$ for which $w \in (U_{\epsilon_0})_*$, so the above equality holds for all $w \in (\Omega_0)_*$. Moreover, $h(w) - \frac{\ln|w|^2}{w} \in C^0(\partial\Omega_0)$ because $0 \notin \partial\Omega_0$, so

$$h(w)-\frac{\ln|w|^2}{w} \doteq C_{\Omega_0}\left[h(w) - \frac{\ln|w|^2}{w}\right] + C_{(\Omega_0)_*}\left[h(w) - \frac{\ln|w|^2}{w}\right] = C_{\Omega_0}\left[h(w) - \frac{\ln|w|^2}{w}\right] \in \mathcal{A}(\Omega_0).$$

That is, there exists a $G \in \mathcal{A}(\Omega_0)$ for which $\frac{\ln|w|^2}{w} \doteq h(w) + G(w) =: S_0(w)$. Hence, Ω_0 admits a generalized Schwarz function S_0, so $\Omega_0 \in \mathrm{QD}_0(h)$ by Theorem 5.1.4. □

Example 5.2.1. We shall see in Theorem 4.5.3 that if $0 \notin \Omega_a$ is an unbounded domain for which $\Omega_a \in \mathrm{QD}_a(\alpha k w^{k-1})$ for some $a, \alpha > 0$ and $k \in \mathbb{Z}_+$, then $\varphi_a(z) = cz\left(1 - \frac{\gamma_k}{z^k}\right)^{\frac{1}{a}}$, where $\gamma_k = -\frac{a\alpha k}{c^{2a-k}}$ is a Riemann map for Ω_a. Then, leaving the details to the reader, the lemma implies that

$$\varphi_0(z) := \lim_{a\to 0^+}\varphi_a(z) = \lim_{a\to 0^+} cz\left(1 + \frac{a}{c^{2a}}\left(\alpha k c^k z^{-k}\right)\right)^{\frac{1}{a}} = cze^{\alpha k c^k z^{-k}}$$

is the Riemann map associated to $\lim_{a\to 0^+}\Omega_a =: \Omega_0 \in \mathrm{QD}_0(\alpha k w^{k-1})$. This result is confirmed by its agreement with the outcome from direct computation (Theorem 5.5.1).

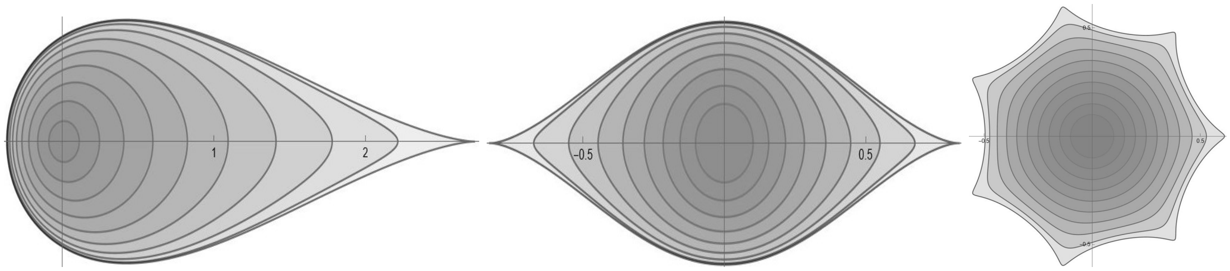

Figure 5.3: Family of domains associated to an unbounded simply connected $\Omega \in \mathrm{QD}_0\left(kw^{k-1}\right)$, $k \in \{1, 2, 7\}$.

5.3 The Inverse and Direct Problems for LQDs

The general results of §2 make it possible to begin addressing the inverse and direct problems for log-weighted quadrature domains in explicit families. The aim of the present section is to analyze the simplest such families, both in order to obtain concrete classification theorems and to illustrate the new geometric features introduced by the singular weight $\rho_0(w) = |w|^{-2}$. The null case, $\mathrm{QD}_0(0)$, provides the most basic test of the theory: the singularity at the origin induces an additional family of null LQDs.

Classification of Null LQDs

An LQD Ω is referred to as a *null LQD* if 0 is its quadrature function: $\Omega \in \mathrm{QD}_0(0)$. We prove the following classification theorem for null LQDs.

Theorem 5.3.1. *A domain is a null LQD if and only if it is a disk or exterior disk centered at the origin.*

Proof of Theorem 5.3.1. Let $\Omega \in \mathrm{QD}_0(0)$. We first note that Ω must contain at least one of $0, \infty$ because otherwise $1 \in L^1_a(\Omega; \rho_0)$ and

$$0 < \int_\Omega |w|^{-2} dA(w) = \oint_{\partial\Omega} 1 \cdot 0 dw = 0.$$

Then, by Equations 5.2 and 5.3, we find that $\ln|w|^2 \doteq wG(w) + q$ for some $q \in \mathbb{C}$ ($q = 0$ if $0 \notin \Omega$) and $G \in \mathcal{A}(\Omega)$.

If Ω is unbounded then $G \in \mathcal{A}_0(\Omega)$ and there exists $q', \widetilde{G} \in \mathcal{A}_0(\Omega)$ such that

$$\ln\left|we^{-q'}\right|^2 \doteq \widetilde{G}(w) \qquad \Longrightarrow \qquad \left|we^{-q'}\right|^2 \doteq e^{\widetilde{G}(w)},$$

where $e^{\widetilde{G}(w)} - 1 \in \mathcal{A}_0(\Omega)$. Hence rearranging the above equation we obtain $U \in \mathcal{A}_0(\Omega)$ and $A \in \mathbb{C}$ for which $|w|^2 \doteq U(w) + A$. Thus if $f \in \mathcal{A}_0(\Omega)$,

$$\int_\Omega f(w) dA(w) = \oint_{\partial\Omega} f(w)\overline{w} dw = \oint_{\partial\Omega} f(w)\frac{U(w) + A}{w} dw.$$

If $0 \notin \Omega$, then this equals 0. On the other hand, if $0 \in \Omega$, then we obtain $f(0)\,(U(0) + A)$. In particular, Ω is either a (classical) null quadrature domain or a (classical) one-point quadrature domain with its quadrature node at $w = 0$.

Aharonov & Shapiro [1] showed that a finitely connected bounded domain is a one-point QD if and only if it is a disk. When combined with Lee & Makarov's [20] topological bounds for QDs, we are able to drop the assumption of finite-connectedness. Hence, if $\Omega \in \mathrm{QD}_0(0)$ is bounded, it is a disk centered at 0.

On the other hand, Sakai [29] showed that a domain with compact boundary is a null QD with respect to the test class $L^1_a(\Omega)$ if and only if it is an exterior disk or an exterior ellipse. However, when considered with respect to the larger test class $A_0(\Omega)$, only exterior disks are null (that ellipses are not follows from direct computation). Hence, if $\Omega \in \mathrm{QD}_0(0)$ is unbounded, it is an exterior disk centered at 0.

For the reverse direction, take $r > 0$ and $f \in L^1_a(\mathbb{D}_r; \rho_0)$, which implies $f(0) = 0$. Integrating against ρ_0 and changing to polar coordinates, we obtain

$$\int_{\mathbb{D}_r} \frac{f(w)}{|w|^2} dA(w) = 2 \int_0^r \oint_{\partial\mathbb{D}} \frac{f(sw)}{sw} dw ds = 0$$

The last equality follows from the fact that $f(0) = 0$, so $\frac{f(sw)}{sw}$ is analytic in $\mathbb{D}$. The argument for $(\mathbb{D}_r)_*$ is entirely analogous. $\square$

Partial Classification of Non-Singular One-Point LQDs

A classic result due to Aharonov & Shapiro [1] states that a finitely connected bounded domain is a one-point classical QD if and only if it is a disk. We prove the following analog of this result for LQDs: A simply connected bounded domain not containing zero is a one-point LQD if and only if it is biholomorphic to a disk via the exponential map $w \mapsto e^w$. In particular, we show the following:

Theorem 5.3.2. *Fix $\alpha, w_0 \in \mathbb{C} \setminus \{0\}$. If Ω is a simply connected bounded domain not containing zero, then $\Omega \in QD_0\left(\frac{\alpha}{w - w_0}\right)$ if and only if $0 < \alpha \leq \pi^2$ and $\Omega = \{w \in \mathbb{C} : |\ln(ww_0^{-1})|^2 < \alpha\}$.*

Moreover, $\partial\Omega$ develops a double point at $\alpha = \pi^2$. Here and throughout this thesis, ln denotes the principal branch of the natural logarithm, with branch cut along the negative real axis.

Proof of Theorem 5.3.2. Beginning with the reverse direction, consider the bounded domain

$$\Omega = \{w \in \mathbb{C} : \left|\ln(ww_0^{-1})\right|^2 < \alpha\}$$

for some $w_0 \neq 0$ and $\alpha > 0$. To solve the direct problem for Ω, we would like to show that $\Omega \in \mathrm{QD}_0$ and determine its quadrature function. We proceed by direct computation of the quadrature identity. Note that if $w \in \partial\Omega$, then $\left|\ln(ww_0^{-1})\right|^2 = \alpha$, so

$$\frac{\ln|w|^2}{w} = \frac{\ln(\overline{ww_0^{-1}}) + \ln(w\overline{w_0})}{w} = \frac{1}{w}\left(\frac{\alpha}{\ln(ww_0^{-1})} + \ln(w\overline{w_0})\right) = \frac{1}{w}\frac{\alpha w_0}{w - w_0} + G(w) =: S_0(w)$$

for some $G \in \mathcal{A}(\Omega)$. S_0 is meromorphic in Ω and extends continuously to the boundary with boundary values $\frac{\ln|w|^2}{w}$. Hence S_0 is a generalized Schwarz function for Ω so, $\Omega \in \mathrm{QD}_0$ by Theorem 5.1.4.

We will now compute the associated quadrature function h directly. Note that $0 \notin \Omega$ so, by Green's theorem and Cauchy's integral theorem, for each $f \in L^1_a(\Omega; \rho_0)$,

$$\begin{aligned}
\int_\Omega \frac{f(w)}{|w|^2} dA(w) &= \oint_{\partial\Omega} f(w) \frac{\ln|w|^2}{w} dw \\
&= \oint_{\partial\Omega} f(w)\left(\frac{1}{w}\frac{\alpha w_0}{w - w_0} + G(w)\right) dw \\
&= \oint_{\partial\Omega} f(w) \frac{\alpha}{w - w_0} dw + \oint_{\partial\Omega} f(w)\left(G(w) - \frac{\alpha}{w}\right) dw \\
&= \oint_{\partial\Omega} f(w) \frac{\alpha}{w - w_0} dw.
\end{aligned}$$

Hence, $\Omega \in \mathrm{QD}_0\left(\frac{\alpha}{w - w_0}\right)$.

For the forward direction, suppose that $\Omega \in \mathrm{QD}_0\left(\frac{\alpha}{w - w_0}\right)$ for some $\alpha, w_0 \in \mathbb{C} \setminus \{0\}$, and that Ω is a bounded simply connected domain not containing zero. We will show that $\alpha > 0$ and $\partial\Omega = \{w \in \mathbb{C} : |\ln(ww_0^{-1})|^2 = \alpha\}$. Firstly, the fact that $\alpha > 0$ follows immediately from applying the quadrature identity to $1 \in L^1_a(\Omega; \rho_0)$:

$$0 < \int_\Omega \frac{dA(w)}{|w|^2} = \oint_{\partial\Omega} \frac{\alpha}{w - w_0} dw = \alpha.$$

By Theorem 5.1.7, $\Omega \in \mathrm{QD}_0\left(\frac{\alpha}{w - w_0}\right)$ if and only if $w_0^{-1}\Omega \in \mathrm{QD}_0\left(\frac{\alpha}{w-1}\right)$. Hence, we can wlog consider the case in which $w_0 = 1$. Next, by Theorem 5.1.2, there exists $G \in \mathcal{A}(\Omega)$ such that $\frac{\ln|w|^2}{w} \doteq \frac{\alpha}{w-1} + G(w)$. Hence,

$$\ln|w|^2 \doteq \frac{w\alpha}{w - 1} + wG(w)$$

As $0 \notin \Omega$ is bounded and simply connected, we know that Ω_* is connected and contains both 0 and ∞. Hence, isolating the simple pole of $\frac{1}{\ln(w)}$ at $w = 1$, we find that $\frac{w\alpha}{w-1} = \frac{\alpha}{\ln(w)} + \widetilde{G}(w)$ for some

$\widetilde{G} \in \mathcal{A}(\Omega)$. Therefore

$$\ln(\overline{w}) + \ln(w) \doteq \frac{\alpha}{\ln(w)} + \widetilde{G}(w) + wG(w)$$

Finally, rearranging, we obtain

$$\overline{\ln(w)} \doteq \frac{\alpha}{\ln(w)} + \widetilde{G}(w) + \ln(w) + wG(w).$$

Hence, $\widehat{G}(w) := \alpha + \ln(w)(\widetilde{G}(w) + \ln(w) + wG(w))$ is a function analytic in Ω for which

$$|\ln(w)|^2 \doteq \widehat{G}(w).$$

Therefore $\mathrm{Im}(\widehat{G}) = 0$ so, by the maximum principle, $\widehat{G}$ is real-valued and analytic in Ω and hence constant on Ω. As $\mathrm{Re}(\widehat{G}) \geq 0$ on $\partial\Omega$, we conclude that $\widehat{G}$ is a non-negative real constant on Ω. In particular, $\exists c \geq 0$ such that $|\ln(w)|^2 \doteq c$. Then, applying the quadrature identity calculation from the first part, we find that $c = \alpha$. In particular, this demonstrates that if $\Omega \in \mathrm{QD}_0\left(\frac{\alpha}{w-1}\right)$, then $\partial\Omega \subseteq \{w \in \mathbb{C} : |\ln(w)|^2 = \alpha\} =: \Gamma_\alpha$.

Finally, we must determine which connected component of Γ_α^c is Ω. Note that $1 \in \Omega$ because 1 is the location of the unique quadrature node. Furthermore, $|\ln(1)|^2 - \alpha = -\alpha < 0$, so Ω must be the connected component of Γ_α^c on which $|\ln(w)|^2 < \alpha$. This concludes the proof of Theorem 5.3.2. □

The Riemann map $\varphi : \mathbb{D} \to \mathbb{C}$ associated to Ω such that $\varphi(0) = w_0$ and $\varphi'(0) > 0$ is given by $\varphi(z) = w_0 e^{z\sqrt{\alpha}}$. We note that φ is univalent in $\mathbb{D}$ if and only if $0 < \alpha \leq \pi^2$, which is precisely the set of α for which $|\ln(ww_0^{-1})|^2 < \alpha$ is simply connected.

5.4 Simply Connected LQDs

This section is primarily concerned with solving the inverse and direct problems for simply connected log-weighted quadrature domains. Recall that the *inverse problem* for log-weighted quadrature domains is concerned with recovering an LQD from its quadrature function. Conversely, the *direct problem* for LQDs is concerned with determining whether a given domain is an LQD and, if so, identifying its quadrature function.

In the simply connected setting, domains are represented by their associated Riemann map. Hence, we may reduce the inverse and direct problems to those of characterizing the Riemann map in terms of the quadrature function and vice-versa. In the case of classical quadrature domains, this scheme is aided by the fact that the associated Riemann map is rational: the inverse and direct problems reduce to the determination of the relationship between the finitely many coefficients of the quadrature function and the finitely many coefficients of the Riemann map.

The situation is more complicated for log-weighted quadrature domains. While the Riemann map associated to a simply connected LQD is not necessarily rational, it turns out that it can still be represented in terms of rational functions. In particular, if φ is the Riemann map associated to a simply connected domain Ω and has the *inner-outer factorization* $\varphi(z) = \varphi_{\text{in}}\varphi_{\text{out}}$ (§2.4) then:

Theorem 5.4.1. *$\Omega \in QD_0$ if and only if φ_{out} extends to the exponential of a rational function.*

As we saw in §2.4, φ_{in} is a product of Blaschke factors, so it is also rational. Combining these facts reduces the inverse and direct problems to the determination of the relationship between the finitely many poles and coefficients of the quadrature function and those of the rational functions comprising the Riemann map.

We are now prepared to prove Theorem 5.4.1, the central result of this section.

Proof of Theorem 5.4.1.
We will begin by considering the **bounded** case: Suppose $\Omega \in \mathrm{QD}_0(h)$ is a simply connected bounded domain with Riemann map $\varphi : \mathbb{D} \to \Omega$. By the definition of QD_0, $0 \notin \partial\Omega$. Hence, by Theorem 2.4.1, $\varphi = \varphi_{\text{in}}\varphi_{\text{out}}$, where $\varphi_{\text{in}} = 1$ when $0 \notin \Omega$, $\varphi_{\text{in}} = b_{z_0}$ when $0 \in \Omega$ (z_0 the unique root of φ in $\mathbb{D}$), and $\varphi_{\text{out}} = Ce^{r^\#}$, where

$$r(z) = \Phi_\varphi^{-1}\left(C_{\Omega_*}\left[\ln|w|^2\right]\right)(z).$$

By the generalized coincidence equation (5.2, 5.3), there exists $G \in \mathcal{A}(\Omega)$ and $q \in \mathbb{C}$[2] such that $\ln|w|^2 \doteq wh(w) + wG(w) + q$. Substituting this into our formula for r, we obtain

$$r = \Phi_\varphi^{-1}\left(C_{\Omega_*}\left[wh(w) + wG(w) + q\right]\right).$$

Next, $wG(w) + q \in \mathcal{A}(\Omega)$, so $C_{\Omega_*}\left[wG(w) + q\right] = 0$. It follows from calculus of residues that

$$C_{\Omega_*}\left[wh(w)\right] = wh(w) + \operatorname*{Res}_{\infty} h,$$

Hence,

$$r = \Phi_\varphi^{-1}\left(wh(w) + \operatorname*{Res}_{\infty} h\right).$$

We conclude that r is rational, as the Faber transform of a rational function.

For the reverse direction, suppose $\varphi_{\text{out}} = e^{r^\#}$ with r rational. Then, as $|\varphi_{\text{in}}| \doteq 1$, we find that

$$\ln|\varphi|^2 \doteq r + r^\#.$$

[2] $q = 0$ if $0 \notin \Omega$.

Hence,

$$S_0(w) := \frac{\left(r + r^{\#}\right) \circ \psi(w)}{w}$$

is meromorphic in Ω, extends continuously to $\partial\Omega$, and satisfies

$$S_0(w) \doteq \frac{\ln |w|^2}{w}.$$

Thus S_0 is a generalized Schwarz function for Ω, and Theorem 5.1.4 yields $\Omega \in \mathrm{QD}_0$.

We will now consider the **unbounded** case: Suppose $\Omega \in \mathrm{QD}_0(h)$ is a simply connected unbounded domain with Riemann map $\varphi : \mathbb{D}_* \to \Omega$. By Theorem 2.4.1, $\varphi = \varphi_{\text{in}}\varphi_{\text{out}}$, where $\varphi_{\text{in}}(z) = z$ when $0 \notin \Omega$, $\varphi_{\text{in}}(z) = zb_{z_0}(z)$ when $0 \in \Omega$ (z_0 the unique root of φ in $\mathbb{D}$), and $\varphi_{\text{out}} = Ce^{r^{\#}}$, where

$$r(z) = \Phi_\varphi^{-1}\left(C_{\Omega_*}\left[\ln |w|^2\right]\right)(z)$$

As $\Omega \in \mathrm{QD}_0(h)$, Theorems 5.1.2 and 5.1.3 tell us there exists $G \in \mathcal{A}_0(\Omega)$ and $q \in \mathbb{C}$ ($q = 0$ if $0 \notin \Omega$) such that $\ln |w|^2 \doteq wh(w) + wG(w) + q$. Substituting this into our formula for r, we obtain

$$r = \Phi_\varphi^{-1}\left(C_{\Omega_*}\left[wh(w) + wG(w) + q\right]\right).$$

By calculus of residues, $C_{\Omega_*}\left[wG(w) + q\right] = q - \operatorname*{Res}_{\infty} G =: q'$. As $wh(w) \in \mathcal{A}(\Omega_*)$, $C_{\Omega_*}\left[wh(w)\right] = wh(w)$. We find that

$$r = \Phi_\varphi^{-1}\left(wh(w)\right) + q'.$$

Hence r is rational, as the Faber transform of a rational function. The argument for the reverse direction is entirely analogous to the argument in the bounded case. □

In view of Theorem 5.4.1, it is convenient to adopt the following normalized factorizations of φ. Under this normalization, φ_{out} is independent of whether $\Omega \in \mathrm{QD}_0(h)$ is singular, and is encoded by a single rational function r.

$$\begin{aligned} \varphi(z) &= w_0 e^{r^{\#}(z)}, && \text{when } 0 \notin \Omega, \\ \varphi(z) &= \frac{w_0}{|z_0|} b_{z_0}(z) e^{r^{\#}(z)}, && \text{when } 0 \in \Omega, \end{aligned} \tag{5.5}$$

where the constant out front is chosen for the normalization $\varphi(0) = w_0 \neq 0$, and $r \in \mathcal{A}_0(\mathbb{D}_*)$. When $\varphi(0) = 0$, we instead take $\varphi(z) = cze^{r^{\#}(z)}$, where $\varphi'(0) = c > 0$, to the same effect. On the other hand, when Ω is unbounded, we write

$$\begin{aligned} \varphi(z) &= cze^{r^{\#}(z)}, && \text{when } 0 \notin \Omega, \\ \varphi(z) &= c|z_0| z b_{z_0}(z) e^{r^{\#}(z)}, && \text{when } 0 \in \Omega, \end{aligned} \tag{5.6}$$

where the constant out front is chosen for the normalization $\varphi'(\infty) = c > 0$ (the conformal radius of Ω), and $r \in \mathcal{A}(\mathbb{D})$, with $r(0) = 0$.

Faber Transform Formulae for Log-Weighted Quadrature Domains

Under the normalizations (5.5, 5.6), Theorem 4.1 reduces the simply connected inverse and direct problems to a single question: how is the rational function r related to the quadrature function h? The next theorem answers this explicitly via the Faber transform.

Theorem 5.4.2. *Let $\Omega \in QD_0(h)$ be a simply connected domain with Riemann map φ.*

1. *If Ω is bounded, then φ admits the factorization 5.5, where r is given by*

$$r(z) = \Phi_\varphi^{-1}\left(wh(w) + \operatorname*{Res}_\infty h\right)(z), \tag{5.7}$$

2. *If Ω is unbounded, then φ admits the factorization 5.6, where r is given by*

$$r(z) = \Phi_\varphi^{-1}\left(wh(w)\right)(z) - \Phi_\varphi^{-1}\left(wh(w)\right)(0). \tag{5.8}$$

Inverting our formulae for r, we obtain a formula for the quadrature function h in terms of r:

$$h(w) = \frac{\Phi_\varphi(r)(w) - C}{w} \tag{5.9}$$

for some $C \in \mathbb{C}$. Note that if $0 \notin \Omega$, then C is determined by the requirement that h has no pole at 0. On the other hand if $0 \in \Omega$ then C is arbitrary (Corollary 5.1.2).

Theorem 5.4.2 gives the explicit relationship between the Riemann map (via r) and the quadrature function h. In particular, once h is known, the formulae above recover r, and hence φ.

Proof of Theorem 5.4.2.
Bounded case: Suppose $\Omega \in \mathrm{QD}_0(h)$ is bounded with Riemann map $\varphi : \mathbb{D} \to \Omega$. By Theorem 2.4.1 and the proof of Theorem 5.4.1, there exists $C \in \mathbb{C}$, and a rational function r satisfying

$$r = \Phi_\varphi^{-1}\left(wh(w) + \operatorname*{Res}_\infty h\right)$$

such that

$$\varphi(z) = C\varphi_{\text{in}}(z)e^{r^\#(z)},$$

where φ_{in} is either 1, a Blaschke factor b_{z_0}, or z, when $0 \notin \Omega$ $\varphi(0) \neq 0$, and $\varphi(0) = 0$ respectively. Since

$$\Phi_\varphi^{-1} : \mathcal{A}_0(\Omega_*) \to \mathcal{A}_0(\mathbb{D}_*)$$

in the bounded case, we have that $r \in \mathcal{A}_0(\mathbb{D}_*)$, hence $r^\# \in \mathcal{A}(\mathbb{D})$ and $r^\#(0) = 0$. Hence the constant C is determined by the normalization at $z = 0$: If $\varphi(0) = w_0 \neq 0$, then

$$C = w_0, \quad \text{or} \quad C = \frac{w_0}{b_{z_0}(0)} = \frac{w_0}{|z_0|},$$

when $0 \notin \Omega$ and $0 \in \Omega$ respectively. If $\varphi(0) = 0$, then the normalization implies

$$\varphi(z) = cze^{r^{\#}(z)},$$

where $c = \varphi'(0) > 0$. Hence φ takes the normalized form given in Equation 5.5.

Unbounded case: Suppose $\Omega \in \mathrm{QD}_0(h)$ is unbounded with Riemann map $\varphi : \mathbb{D}_* \to \Omega$ normalized by $\varphi(\infty) = \infty$ and $\varphi'(\infty) = c > 0$. By Theorem 2.4.1 and the proof of Theorem 5.4.1, there exists $C \in \mathbb{C}$, and a rational function $\widetilde{r}$ such that

$$\widetilde{r} = \Phi_{\varphi}^{-1}\left(wh(w)\right) + q'$$

for some $q' \in \mathbb{C}$, and

$$\varphi(z) = C\varphi_{\text{in}}(z)e^{\widetilde{r}^{\#}(z)},$$

where φ_{in} is either z or $zb_{z_0}(z)$, when $0 \notin \Omega$ and $0 \in \Omega$ respectively. Since

$$\Phi_{\varphi}^{-1} : \mathcal{A}(\Omega_*) \to \mathcal{A}(\mathbb{D})$$

in the unbounded case, we have that $\widetilde{r} \in \mathcal{A}(\mathbb{D})$. Subtracting the constant $\widetilde{r}(0)$, we obtain a rational function

$$r(z) = \widetilde{r}(z) - \widetilde{r}(0) = \Phi_{\varphi}^{-1}\left(wh(w)\right)(z) - \Phi_{\varphi}^{-1}\left(wh(w)\right)(0),$$

so that $r(0) = 0$. Absorbing the corresponding multiplicative constant in C and imposing the normalization $\varphi'(\infty) = c > 0$, we obtain the precise forms of Equation 5.6. $\square$

Theorem 5.4.2 and Equations 5.7, 5.8, and 5.9 will be used extensively throughout the remainder of the manuscript. In particular, we will apply these formulae to address the inverse and direct problems for several different classes of simply connected LQDs.

Example: Constant Log-Weighted Quadrature Domains

In this section, we provide a classification of simply connected "constant" LQDs: $\Omega \in \mathrm{QD}_0(\alpha)$, $\alpha \in \mathbb{C}$. Note that Ω must be unbounded because ∞ is a quadrature node, so there are two cases to consider: (1) $0 \notin \Omega$ and (2) $0 \in \Omega$. Let $\varphi : \mathbb{D}_* \to \Omega$ be the associated Riemann map. Graven and Makarov [13] demonstrated that if $0 \notin \Omega$, then $\varphi(z) = cze^{\overline{\alpha}cz^{-1}}$, where $0 < c < |\alpha|^{-1}$ is the conformal radius of Ω. Thus we will restrict our attention to the case in which $0 \in \Omega$.

Theorem 5.4.3. *Fix $\alpha \neq 0$ and a simply connected domain Ω containing zero. Then $\Omega \in QD_0(\alpha)$ if and only if it admits a Riemann map of the form*

$$\varphi(z) = c|z_0|zb_{z_0}(z)e^{\overline{\alpha}cz^{-1}}. \tag{5.10}$$

Proof. By Equation 5.6, we know that $\varphi(z) = c|z_0|zb_{z_0}(z)e^{r^{\#}(z)}$, where z_0 is the unique root of φ. Moreover, by Equation 5.8, $r(z) = \Phi_\varphi^{-1}(w\alpha)(z) - \Phi_\varphi^{-1}(w\alpha)(0)$. By the linearity of the Faber transform and Equation 2.11, we obtain

$$r(z) = \alpha W_1(z) - \alpha W_1(0) = \alpha cz + \alpha f_0 - \alpha f_0,$$

where $W_j(z) = \Phi_\varphi^{-1}(w^j)$ is the jth inverse Faber polynomial (§2.3). Hence,

$$\varphi(z) = c|z_0|zb_{z_0}(z)e^{\overline{\alpha}cz^{-1}}.$$

In particular we have shown that if $0 \in \Omega \in \mathrm{QD}_0(\alpha)$ is simply connected, then it admits a Riemann map of the above form (Equation 5.10). On the other hand, by Theorem 5.4.1, we know that the image Ω of the above φ is in $\mathrm{QD}_0(h)$ for some h. Hence, as Ω is unbounded, Equation 5.9 tells us there exists $C \in \mathbb{C}$ such that

$$h(w) = \frac{\Phi_\varphi(\alpha cz)(w) - C}{w} = \frac{\alpha c\left(\frac{w}{c} - \frac{f_0}{c}\right) - C}{w} = \alpha - \frac{\alpha f_0 + C}{w}$$

Thus $\Omega \in \mathrm{QD}_0\left(\alpha + \frac{q}{w}\right)$ for some $q \in \mathbb{C}$. We conclude by Corollary 5.1.2 that $\Omega \in \mathrm{QD}_0(\alpha)$. □

This is only a special case of the class of *monomial* LQDs, which is addressed in the next section.

5.5 Monomial Log-Weighted Quadrature Domains

In this section, we consider simply connected LQDs with quadrature function $h(w) = \alpha k w^{k-1}$ for $k \in \mathbb{Z}_+$ and $\alpha \in \mathbb{C} \setminus \{0\}$. Note that all such domains are unbounded, as ∞ is a quadrature node. When Ω is non-singular, we are able to construct simple solutions by exploiting the symmetry of the quadrature function. As we shall see, no such symmetry is possible when Ω is singular, resulting in a much larger space of possible solutions. Hence, we only construct explicit solutions in the singular case for $k = 2$ (The $k = 1$ case was covered in §5.4).

Symmetric Non-Singular Monomial LQDs

Suppose that $\Omega \in \mathrm{QD}_0(\alpha k w^{k-1})$ is a simply connected domain not containing zero. As Ω is unbounded, Theorem 5.4.2 tells us that the associated Riemann map $\varphi : \mathbb{D}_* \to \Omega$ is of the form $\varphi(z) = cze^{r^{\#}(z)}$, where $c > 0$ is the conformal radius of Ω and

$$r(z) = \Phi_\varphi^{-1}\left(\alpha k w^k\right)(z) - \Phi_\varphi^{-1}\left(\alpha k w^k\right)(0).$$

We recognize $\Phi_\varphi^{-1}\left(\alpha k w^k\right)(z) = \alpha k \Phi_\varphi^{-1}\left(w^k\right)(z) = \alpha k W_k(z)$, where W_k is the kth inverse Faber polynomial (§2.3). Hence, $r(z) = \alpha k\left(W_k(z) - W_k(0)\right)$, and we find

$$\varphi(z) = cze^{\overline{\alpha}k\left(W_k^{\#}(z) - \overline{W_k(0)}\right)}.$$

We now consider the implications of the symmetry of the quadrature function. By Theorem 5.1.7, for each $j \in \{1, \ldots, k\}$, we have $\Omega \in \mathrm{QD}_0(\alpha k w^{k-1})$ if and only if $e^{-2\pi i \frac{j}{k}}\Omega \in \mathrm{QD}_0(\alpha k w^{k-1})$. That is, $\Omega \in \mathrm{QD}_0(\alpha k w^{k-1})$ implies each rotation of Ω by a multiple of $\frac{2\pi}{k}$ radians is an LQD with the same quadrature function. Hence, if $\Omega \in \mathrm{QD}_0(\alpha k w^{k-1})$ is the unique such domain of its conformal radius, then it has $k-$fold rotational symmetry about the origin. It follows via inspection of the Laurent expansion that if a simply connected domain has $k-$fold rotational symmetry about the origin, then so does its Riemann map: $\varphi\left(z e^{\frac{2\pi i}{k}}\right) = e^{\frac{2\pi i}{k}}\varphi(z)$. Hence, $W_k^{\#}\left(z e^{\frac{2\pi i}{k}}\right) - \overline{W_k(0)} = e^{\frac{2\pi i}{k}}\left(W_k^{\#}(z) - \overline{W_k(0)}\right)$. Iterating this identity, we find that each of the coefficients of $W_k(z)$ must equal zero except for the term of order k, $c^k z^k$. That is,

$$\varphi(z) = cze^{\overline{\alpha}kc^kz^{-k}} \tag{5.11}$$

In summary,

Theorem 5.5.1. *If $\Omega \in QD_0(\alpha k w^{k-1})$ is a simply connected domain not containing zero which is $k-$fold rotationally symmetric about the origin, then its associated Riemann map is given by Equation 5.11.*

Furthermore, note that this family is univalent precisely when $0 < c < |\alpha k^2|^{-\frac{1}{k}}$. A similar analysis is carried out in Graven & Makarov [13].

Singular Monomial LQDs

Now suppose that $\Omega \in \mathrm{QD}_0(\alpha k w^{k-1})$ is a simply connected domain containing zero. As Ω is unbounded, Theorem 5.4.2 tells us that the associated Riemann map $\varphi : \mathbb{D}_* \to \Omega$ is of the form $\varphi(z) = c|z_0| z b_{z_0}(z) e^{r^{\#}(z)}$, where $c > 0$ is the conformal radius of Ω, $z_0 \in \mathbb{D}_*$ is the unique root of φ, and r is given by Equation 5.8. By the same argument as in the non-singular case, we find that

$$\varphi(z) = c|z_0| z b_{z_0}(z) e^{\overline{\alpha}k\left(W_k^{\#}(z) - \overline{W_k(0)}\right)}. \tag{5.12}$$

We note that Ω cannot be $k-$fold rotationally symmetric: If Ω were, then it must either be multiply connected, or equal the whole plane, because it contains an open neighborhood of both zero and infinity. As we assumed that Ω is connected, we would conclude that $\Omega = \widehat{\mathbb{C}}$, which is not conformally equivalent to $\mathbb{D}_*$.

It remains to determine a relation involving the charge q at the origin under the specification $\Omega \in \mathrm{QD}_0\left(\alpha k w^{k-1}; q\right)$. We begin by computing the generalized Schwarz function. Note that $\frac{\ln|w|^2}{w} \doteq \frac{\ln(\varphi\varphi^{\#})\circ\psi(w)}{w}$, so

$$S_0(w) := \frac{\ln(\varphi\varphi^{\#}) \circ \psi(w)}{w} = \frac{\ln|cz_0|^2 + \overline{\alpha}k\left(W_k^{\#} \circ \psi(w) - \overline{W_k(0)}\right) + \alpha k\left(W_k \circ \psi(w) - W_k(0)\right)}{w}$$

Applying the fact that $W_k \circ \psi(w) = w^k + O(1)$, we find that

$$S_0(w) = \alpha k w^{k-1} + \frac{\ln |cz_0|^2 - \alpha k W_k(0)}{w} + O(w^{-1}).$$

Hence, $q = \ln |cz_0|^2 - \alpha k W_k(0)$.

Example: The case of k=2

We consider the $k = 2$ case as a basic example. As $W_2(z)$ is a polynomial of degree 2 with leading term $c^2 z^2$, we find that

$$\varphi(z) = c|z_0| z b_{z_0}(z) e^{2\overline{\alpha}\left(c^2 z^{-2} + \beta z^{-1}\right)},$$

for some $\beta \in \mathbb{C}$. Applying Equation 5.9 yields

$$\begin{aligned} 2\alpha w &= \frac{\Phi_\varphi\left(2\alpha(c^2 z^2 + \overline{\beta} z)\right)(w) - C}{w} \\ &= \frac{2\alpha c^2}{w} F_2(w) + \frac{2\alpha\overline{\beta}}{w} F_1(w) - \frac{C}{w}, \end{aligned}$$

where F_1 and F_2 are the first and second Faber polynomials associated to φ. Computing these, and substituting, we obtain

$$2\alpha w = 2\alpha w + \frac{2\alpha}{c\overline{z_0}}\left(\overline{\beta}\overline{z_0} - 4c^2\overline{\alpha}\beta\overline{z_0} + 2c^2(|z_0|^2 - 1)\right) + O(1)$$

As the second term must $= 0$, we find that

$$\beta = \frac{2c^2\left(|z_0|^2 - 1\right)\left(4c^2\alpha z_0 + \overline{z_0}\right)}{|z_0|^2\left(16c^4|\alpha|^2 - 1\right)}.$$

Figure 5.4 shows two different families of solutions obtained using this method.

5.6 One-Point Log-Weighted Quadrature Domains

Let Ω be a simply connected LQD with quadrature function $\frac{\alpha}{w-w_0}$ for some $\alpha \in \mathbb{C} \setminus \{0\}$ and $w_0 \in \mathbb{C}$; that is, $\Omega \in \mathrm{QD}_0\left(\frac{\alpha}{w-w_0}\right)$. If $w_0 = 0$ then Ω is a disk centered at the origin. This follows from Corollary 5.1.2, which tells us that $\mathrm{QD}_0\left(\frac{\alpha}{w}\right) = \{\Omega \in \mathrm{QD}_0(0) : 0 \in \Omega\}$, and Theorem 5.3.1, which tells us that the only null quadrature domains containing 0 are disks centered at the origin. Hence, we will assume that $w_0 \neq 0$. Furthermore, by Theorem 5.1.7, $\Omega \in \left(\frac{\alpha}{w-w_0}\right)$ if and only if $w_0^{-1}\Omega \in \left(\frac{\alpha}{w-1}\right)$. Hence, we may wlog assume that $w_0 = 1$.

The behavior of a given family of solutions depends crucially on the singularities of ρ_0 therein. If $0 \in \Omega$, then the choice of charge $q \in \mathbb{C}$ at 0 serves as a free parameter. Moreover, when $\infty \in \Omega$,

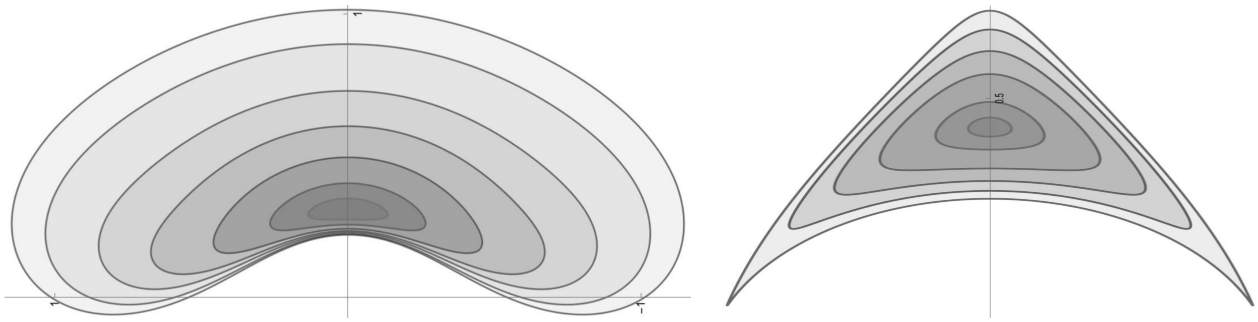

Figure 5.4: Family of complements of singular monomial LQDs in $\mathrm{QD}_0(2\alpha w)$. For $\alpha = -2$, $q = -2$, $0 < c \le .3$ (left); and $\alpha = 1$, $q = -2$, $0 < c < .25$ (right). The plot is rotated so the positive real axis points upwards.

the conformal radius c of the domain serves as an additional free parameter. That is, for a given family of solutions, *there are as many free parameters as there are singularities* of ρ_0 in Ω. Thus it is natural to break up the classification of simply connected $\Omega \in \mathrm{QD}_0\left(\frac{\alpha}{w-w_0}\right)$ into four cases:

1. Non-Singular ($0 \notin \Omega$):
   a) Ω bounded.
   b) Ω unbounded.
2. Singular ($0 \in \Omega$):
   a) Ω bounded.
   b) Ω unbounded.

Non-Singular One-Point LQDs

Let us begin by considering the more tractable case of non-singular one-point LQDs. In particular, we consider simply connected domains $\Omega \in \mathrm{QD}_0\left(\frac{\alpha}{w-w_0}\right)$ for $\alpha, w_0 \in \mathbb{C} \setminus \{0\}$ not containing zero. The bounded case was addressed in §5.3, so we shall restrict our consideration to unbounded Ω, the results of which are summarized in Theorem 5.6.1.

Theorem 5.6.1. *Take $w_0, \alpha \in \mathbb{C} \setminus \{0\}$. There exists an unbounded simply connected domain Ω not containing zero for which $\Omega \in QD_0\left(\frac{\alpha}{w-w_0}\right)$ if and only if there exist $c > 0$, $\lambda \in \mathbb{C}$, and $z_1 \in \mathbb{D}_*$ for which $\Omega = \varphi(\mathbb{D}_*)$, where φ is univalent and given by*

$$\varphi(z) = cze^{\frac{\lambda}{1-z\overline{z_1}}}, \tag{5.13}$$

where $\varphi(z_1) = w_0$, and the coefficients satisfy the relation $\lambda = \frac{\overline{\alpha w_0}}{\overline{z_1}\overline{\varphi'(z_1)}}$.

Proof of Theorem 5.6.1. For the forward direction, suppose that $0 \notin \Omega \in \mathrm{QD}_0\left(\frac{\alpha}{w-1}\right)$ with $\alpha \in \mathbb{C} \setminus \{0\}$ is unbounded and simply connected with Riemann map $\varphi : \mathbb{D}_* \to \Omega$, with $\varphi(z_1) = 1$, and $\varphi'(\infty) = c > 0$. By Theorem 5.4.2, $\varphi(z) = cze^{r^{\#}(z)}$, where $r(z) = \Phi_\varphi^{-1}\left(\frac{\alpha w}{w-1}\right)(z) - \Phi_\varphi^{-1}\left(\frac{\alpha w}{w-1}\right)(0)$. Computing the Faber transform via Equation 2.11, we find

$$\Phi_\varphi^{-1}\left(\frac{\alpha w}{w-1}\right)(z) = \alpha + \alpha\Phi_\varphi^{-1}\left(\frac{1}{w-1}\right)(z) = \alpha\left(1 + \frac{\psi'(1)}{z - \psi(1)}\right)$$

so $r(z) = \frac{\alpha}{z_1\varphi'(z_1)}\frac{z}{z-z_1}$, and we recover Equation 5.13 by setting $\lambda = \frac{\overline{\alpha}}{\overline{z_1\varphi'(z_1)}}$. Also, $z_1 \in \mathbb{D}_*$ because otherwise φ wouldn't be analytic.

For the reverse direction, suppose that there exist $c > 0$, $\lambda \in \mathbb{C}$, and $z_1 \in \mathbb{D}_*$ for which $\Omega = \varphi(\mathbb{D}_*)$, where φ is univalent and given by Equation 5.13. Then, by Theorem 5.4.1, $\Omega \in \mathrm{QD}_0(h)$ for some h. We will now compute h using Theorem 5.4.2, which tells us that there exists $C \in \mathbb{C}$ such that

$$h(w) = \frac{\Phi_\varphi\left(\frac{\overline{\lambda}z}{z-z_1}\right)(w) - C}{w} = \frac{z_1\overline{\lambda}}{w}\Phi_\varphi\left(\frac{1}{z-z_1}\right)(w) + \frac{\overline{\lambda} - C}{w}.$$

Computing the Faber transform via Equation 2.11 and substituting the formula for λ, we find

$$h(w) = \frac{z_1\overline{\lambda}}{w}\frac{\varphi'(z_1)}{w - \varphi(z_1)} + \frac{\overline{\lambda} - C}{w} = \frac{\alpha}{w-1} + \frac{\overline{\lambda} - C - \alpha}{w}.$$

Finally, as $h \in \mathcal{A}(\Omega_*)$ and $0 \in \Omega_*$ ($\varphi(z) = 0$ has no solutions in $\mathbb{D}_*$, so $0 \notin \Omega$), we find that $C = \overline{\lambda} - \alpha$, so $\Omega \in \mathrm{QD}_0\left(\frac{\alpha}{w-1}\right)$. □

Singular One-Point LQDs

We now turn our consideration to the more exotic case of singular one-point LQDs. In particular, we consider simply connected domains $\Omega \in \mathrm{QD}_0\left(\frac{\alpha}{w-w_0}\right)$ which contain zero (recall that we can wlog take $w_0 = 1$).

Singular Bounded One-Point LQDs

Theorem 5.6.2. *Take $w_0, \alpha \in \mathbb{C} \setminus \{0\}$. There exists a bounded simply connected domain Ω containing zero for which $\Omega \in QD_0\left(\frac{\alpha}{w-w_0}\right)$ if and only if there are $\lambda \in \mathbb{C}$ and $z_0 \in \mathbb{D}$ for which $\Omega = \varphi(\mathbb{D})$, where φ is univalent and given by*

$$\varphi(z) = \frac{w_0}{|z_0|}b_{z_0}(z)e^{\lambda z}, \tag{5.14}$$

where $\lambda = \frac{\overline{w_0\alpha}}{\varphi'(0)}$. Moreover, if we specify that $\Omega \in QD_0\left(\frac{\alpha}{w-w_0}; q\right)$, then $\lambda = \overline{w_0\alpha}\frac{q+\ln|z_0|^2}{\overline{\alpha}z_0+\alpha z_0^{-1}}$.

Here, φ is normalized so that $\varphi(0) = w_0$ and $\varphi'(0) > 0$.

Proof of Theorem 5.6.2. For the forward direction, suppose that $0 \in \Omega \in \mathrm{QD}_0\left(\frac{\alpha}{w-1}\right)$ with $\alpha \in \mathbb{C} \setminus \{0\}$ is bounded and simply connected with Riemann map $\varphi : \mathbb{D} \to \Omega$, normalized such that $\varphi(z_0) = 0$, $\varphi(0) = 1$, and $\varphi'(0) > 0$. Note that, unlike in the non-singular case (§5.3), we cannot assume $\alpha > 0$ because $1 \notin L^1_a(\Omega; \rho_0)$ when $0 \in \Omega$.

By Theorem 5.4.2, $\varphi(z) = \frac{1}{|z_0|} b_{z_0}(z) e^{r^\#(z)}$, where $r(z) = \Phi_\varphi^{-1}\left(\frac{\alpha w}{w-1} - \alpha\right)(z) = \alpha \Phi_\varphi^{-1}\left(\frac{1}{w-1}\right)(z)$. Computing the Faber transform using Equation 2.11, we obtain $r(z) = \alpha \frac{\psi'(1)}{z - \psi(1)} = z^{-1} \frac{\alpha}{\varphi'(0)}$. Hence, setting $\lambda = \frac{\overline{\alpha}}{\varphi'(0)}$, we obtain

$$\varphi(z) = \frac{1}{|z_0|} b_{z_0}(z) e^{\lambda z}.$$

For the reverse direction, suppose there exist $\lambda \in \mathbb{C}$ and $z_0 \in \mathbb{D}$ for which $\Omega = \varphi(\mathbb{D})$, where φ is univalent and given by Equation 5.14 (with $w_0 = 1$). Then, by Theorem 5.4.1, $\Omega \in \mathrm{QD}_0(h)$ for some h. We will now compute h using Theorem 5.4.2, which tells us that there exists $C \in \mathbb{C}$ such that

$$h(w) = \frac{\Phi_\varphi\left(\overline{\lambda} z^{-1}\right)(w) - C}{w} = \frac{\overline{\lambda}}{w} \Phi_\varphi\left(z^{-1}\right)(w) - \frac{C}{w}$$

Computing the Faber transform via Equation 2.11 and substituting the formula for λ, we find

$$h(w) = \frac{\overline{\lambda}}{w} \frac{\varphi'(0)}{w - \varphi(0)} - \frac{C}{w} = \frac{\alpha}{w-1} + \frac{\alpha - C}{w}.$$

Hence $\Omega \in \mathrm{QD}_0\left(\frac{\alpha}{w-1}\right)$ by Corollary 5.1.2.

It remains to show the formula for λ in terms of q under the specification $\Omega \in \mathrm{QD}_0\left(\frac{\alpha}{w-1}; q\right)$. We begin by computing the generalized Schwarz function. Note that $\frac{\ln|w|^2}{w} \doteq \frac{\ln(\varphi\varphi^\#)\circ\psi(w)}{w}$, so

$$S_0(w) := \frac{\ln(\varphi\varphi^\#) \circ \psi(w)}{w} = -\frac{\ln|z_0|^2}{w} + \frac{1}{\varphi'(0)} \frac{\overline{\alpha}\psi(w) + \alpha\psi(w)^{-1}}{w}$$

is a generalized Schwarz function for Ω. Noting the singularities at $w = 0$ and $w = 1$, computing their residues, and applying Mittag-Leffler's theorem, we find that there exists $G \in \mathcal{A}(\Omega)$ such that

$$S_0(w) = \frac{\frac{\alpha}{\varphi'(0)^2 z_0}\left(z_0\overline{\alpha} + \varphi'(0)(|z_0|^2 - 1)\right)}{w-1} + \frac{\frac{\overline{\alpha} z_0 + \alpha z_0^{-1}}{\varphi'(0)} - \ln|z_0|^2}{w} + G(w).$$

Hence, $q = \frac{\overline{\alpha} z_0 + \alpha z_0^{-1}}{\varphi'(0)} - \ln|z_0|^2$. We conclude that $\lambda = \overline{\alpha} \frac{q + \ln|z_0|^2}{\overline{\alpha} z_0 + \alpha z_0^{-1}}$.

We then obtain the formulae for arbitrary w_0 by replacing instances of φ and φ' with $w_0^{-1}\varphi$ and $w_0^{-1}\varphi'$ respectively. $\square$

Hence, determining the region $\Omega \in \mathrm{QD}_0\left(\frac{\alpha}{w-1}; q\right)$ for each value of q has been reduced to determining z_0. This can be obtained directly via the relation $\varphi'(0) = \frac{\overline{\alpha}}{\lambda}$, which yields after simplification $(1 - |z_0|^2)\lambda = (|\lambda|^2 - \overline{\alpha}) z_0$.

Univalence criteria We would like to characterize the set of values of z_0 and λ for which $\varphi(z) = \frac{1}{|z_0|} b_{z_0}(z) e^{\lambda z}$ is univalent in $\mathbb{D}$. Notice that if we set $f(z) := \varphi\left(\frac{|\lambda|}{\lambda} z\right) = \frac{1}{|z_1|} b_{z_1}(z) e^{|\lambda| z}$, where $z_1 = \frac{\lambda}{|\lambda|} z_0$, then φ is univalent in $\mathbb{D}$ if and only if f is univalent in $\mathbb{D}$. In particular, this shows that it is sufficient to consider φ for $\lambda > 0$.

Note that $|z_1| < 1$ is necessary because otherwise f is not analytic in $\mathbb{D}$. This is also a sufficient condition for f to be analytic in $\mathbb{D}$. Hence, it is necessary and sufficient that $|z_1| < 1$ and $f(\partial\mathbb{D})$ has no self-intersections. Suppose there existed $z \neq w \in \partial\mathbb{D}$ such that $f(z) = f(w)$. Taking the modulus of both sides and recalling that $|b_{z_1}(z)| = 1$, we find $e^{|\lambda|\mathrm{Re}(z)} = e^{|\lambda|\mathrm{Re}(w)}$, which implies $\mathrm{Re}(z) = \mathrm{Re}(w)$. It follows that $w = z^{-1}$, so there exist $z \neq w \in \partial\mathbb{D}$ such that $f(z) = f(w)$ if and only if there exists $z \in \partial\mathbb{D} \cap \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ such that $f(z) = f(z^{-1})$. Rearranging, we find f is univalent in $\mathbb{D}$ if and only if the following equation has no solutions $z \in \partial\mathbb{D} \cap \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.

$$e^{2i|\lambda|\mathrm{Im}(z)} = \frac{(zz_1 - 1)(z\overline{z_1} - 1)}{(z - \overline{z_1})(z - z_1)}.$$

Note that $\frac{(zz_1-1)(z\overline{z_1}-1)}{(z-\overline{z_1})(z-z_1)} = \frac{\left(\overline{z^{-1}(z-\overline{z_1})(z-z_1)}\right)}{z^{-1}(z-\overline{z_1})(z-z_1)} = e^{-2i\mathrm{Arg}(z^{-1}(z-z_1)(z-\overline{z_1}))}$. Hence, setting $z = e^{i\theta}$ $(0 < \theta < \pi)$, and $z_1 = \frac{\lambda}{|\lambda|} z_0 = |z_0| e^{i(\mathrm{Arg}(z_0) + \mathrm{Arg}(\lambda))}$, we find $e^{2i(|\lambda|\sin(\theta) + \delta(\theta))} = 1$, where

$$\delta(\theta) = \mathrm{Arg}\left((1 + |z_0|^2)\cos(\theta) - 2|z_0|\cos(\mathrm{Arg}(z_0) + \mathrm{Arg}(\lambda)) + i(1 - |z_0|^2)\sin(\theta)\right) \tag{5.15}$$

As $\delta(\theta) \in (0, \pi)$ and $|\lambda|\sin(\theta) \in (0, |\lambda|)$ when $0 < \theta < \pi$, we find that if there is a solution to $e^{2i(|\lambda|\sin(\theta)+\delta(\theta))} = 1$, there must be a solution such that $|\lambda|\sin(\theta) + \delta(\theta) = \pi$. Hence, if we set

$$\lambda_{\max} = \min_{0<\theta<\pi} \frac{\pi - \delta(\theta)}{\sin(\theta)}, \tag{5.16}$$

then f is univalent in $\mathbb{D}$ if and only if $0 < |\lambda| \leq \lambda_{\max}$. In particular, we have shown that

Corollary 5.6.1. $\varphi : \mathbb{D} \to \mathbb{C}$ from Equation 5.14 is univalent if and only if $|z_0| < 1$, and $0 < |\lambda| \leq \lambda_{\max}$.

Singular Unbounded One-Point LQDs

Theorem 5.6.3. *Take $w_0, \alpha \in \mathbb{C} \setminus \{0\}$. There exists an unbounded simply connected domain Ω containing zero for which $\Omega \in QD_0\left(\frac{\alpha}{w - w_0}\right)$ if and only if there exist $\lambda \in \mathbb{C}$ and $z_0, z_1 \in \mathbb{D}_*$, for which $\Omega = \varphi(\mathbb{D}_*)$, where φ is univalent and given by*

$$\varphi(z) = c|z_0| z b_{z_0}(z) e^{\frac{\lambda}{1 - z\overline{z_1}}}, \tag{5.17}$$

where $\varphi(z_1) = w_0$ and $\lambda = \frac{\overline{\alpha w_0}}{\overline{z_1}\varphi'(z_1)}$. Moreover, if $\Omega \in QD_0\left(\frac{\alpha}{w - w_0}; q\right)$, then $q = \frac{\lambda}{1 - z_0\overline{z_1}} + \frac{\overline{\lambda}}{1 - z_0^{-1} z_1} + \ln|cz_0|^2$.

Proof of Theorem 5.6.3. Recall that, by Theorem 5.1.7, $\Omega \in \left(\frac{\alpha}{w-w_0}\right)$ if and only if $w_0^{-1}\Omega \in \left(\frac{\alpha}{w-1}\right)$, so it is sufficient to consider the case in which $w_0 = 1$.

For the forward direction, suppose that $0 \in \Omega \in \mathrm{QD}_0\left(\frac{\alpha}{w-1}\right)$ with $\alpha \in \mathbb{C} \setminus \{0\}$ is unbounded and simply connected with Riemann map $\varphi : \mathbb{D}_* \to \Omega$, normalized such that $\varphi(z_0) = 0$, $\varphi(z_1) = 1$, and $\varphi'(\infty) = c > 0$. By Theorem 5.4.2, $\varphi(z) = c|z_0|zb_{z_0}(z)e^{r^\#(z)}$, where $r(z) = \Phi_\varphi^{-1}\left(\frac{\alpha w}{w-1}\right)(z) - \Phi_\varphi^{-1}\left(\frac{\alpha w}{w-1}\right)(0)$. Computing the Faber transform using Equation 2.11, we obtain

$$\Phi_\varphi^{-1}\left(\frac{\alpha w}{w-1}\right)(z) = \alpha + \alpha\Phi_\varphi^{-1}\left(\frac{1}{w-1}\right)(z) = \alpha\left(1 + \frac{\psi'(1)}{z-\psi(1)}\right) = \alpha\left(1 + \frac{1}{\varphi'(z_1)}\frac{1}{z-z_1}\right).$$

Thus $r(z) = \frac{\alpha}{z_1\varphi'(z_1)}\frac{z}{z-z_1}$ and, setting $\lambda = \frac{\overline{\alpha}}{\overline{z_1\varphi'(z_1)}}$, we find

$$\varphi(z) = c|z_0|zb_{z_0}(z)e^{\frac{\lambda}{1-z\overline{z_1}}}$$

For the reverse direction, suppose there exist $\lambda \in \mathbb{C}$ and $z_0, z_1 \in \mathbb{D}_*$ for which $\Omega = \varphi(\mathbb{D}_*)$, where φ is univalent and given by Equation 5.17 (with $w_0 = 1$). Then, by Theorem 5.4.1, $\Omega \in \mathrm{QD}_0(h)$ for some h. We will now compute h using Theorem 5.4.2, which tells us that there exists $C \in \mathbb{C}$ such that

$$h(w) = \frac{\Phi_\varphi\left(\frac{\overline{\lambda}z}{z-z_1}\right)(w) - C}{w} = \frac{\overline{\lambda}z_1}{w}\Phi_\varphi\left(\frac{1}{z-z_1}\right)(w) + \frac{\overline{\lambda} - C}{w}$$

Computing the Faber transform via Equation 2.11 and substituting the formula for λ, we find

$$h(w) = \frac{\overline{\lambda}z_1}{w}\frac{\varphi'(z_1)}{w-\varphi(z_1)} + \frac{\overline{\lambda}-C}{w} = \frac{\alpha}{w-1} + \frac{\overline{\lambda}-C-\alpha}{w}.$$

Hence $\Omega \in \mathrm{QD}_0\left(\frac{\alpha}{w-1}\right)$ by Corollary 5.1.2.

It remains to show the formula for q under the specification $\Omega \in \mathrm{QD}_0\left(\frac{\alpha}{w-1}; q\right)$. We begin by computing the generalized Schwarz function. Note that $\frac{\ln|w|^2}{w} \doteq \frac{\ln(\varphi\varphi^\#)\circ\psi(w)}{w}$, so

$$S_0(w) := \frac{\ln(\varphi\varphi^\#)\circ\psi(w)}{w} = \frac{\ln|cz_0|^2 + \left(\frac{\lambda}{1-\psi(w)\overline{z_1}} + \frac{\overline{\lambda}\psi(w)}{\psi(w)-z_1}\right)}{w}$$

is a generalized Schwarz function for Ω. Noting the singularities at $w = 0$ and $w = 1$, computing the residues, and applying Mittag-Leffler's theorem, we find that there exist $\widetilde{C} \in \mathbb{C}$ and $G \in \mathcal{A}_0(\Omega)$ such that

$$S_0(w) = \frac{\widetilde{C}}{w-1} + \frac{\frac{\lambda}{1-z_0\overline{z_1}} + \frac{\overline{\lambda}}{1-z_0^{-1}z_1} + \ln|cz_0|^2}{w} + G(w).$$

Hence, $q = \frac{\lambda}{1-z_0\overline{z_1}} + \frac{\overline{\lambda}}{1-z_0^{-1}z_1} + \ln|cz_0|^2$.

We then recover formulae for arbitrary w_0 by replacing instances of φ and φ' with $w_0^{-1}\varphi$ and $w_0^{-1}\varphi'$ respectively. □

Note that both Ω and Ω^{-1} are unbounded domains containing zero. Moreover, by Corollary 5.1.4, $\Omega \in \mathrm{QD}_0\left(\frac{\alpha}{w-w_0}; q\right)$ if and only if $\Omega^{-1} \in \mathrm{QD}_0\left(\frac{\alpha}{w-w_0^{-1}} - \frac{\alpha}{w}; -q - A_{\rho_0}(\Omega_*)\right)$.

$$\Omega^{-1} \in \mathrm{QD}_0\left(\frac{\alpha}{w-w_0^{-1}}; -q - \alpha - A_{\rho_0}(\Omega_*)\right).$$

Setting $w_0 = 1$, we find that Ω and Ω^{-1} have the same quadrature function, simply for different values of q.

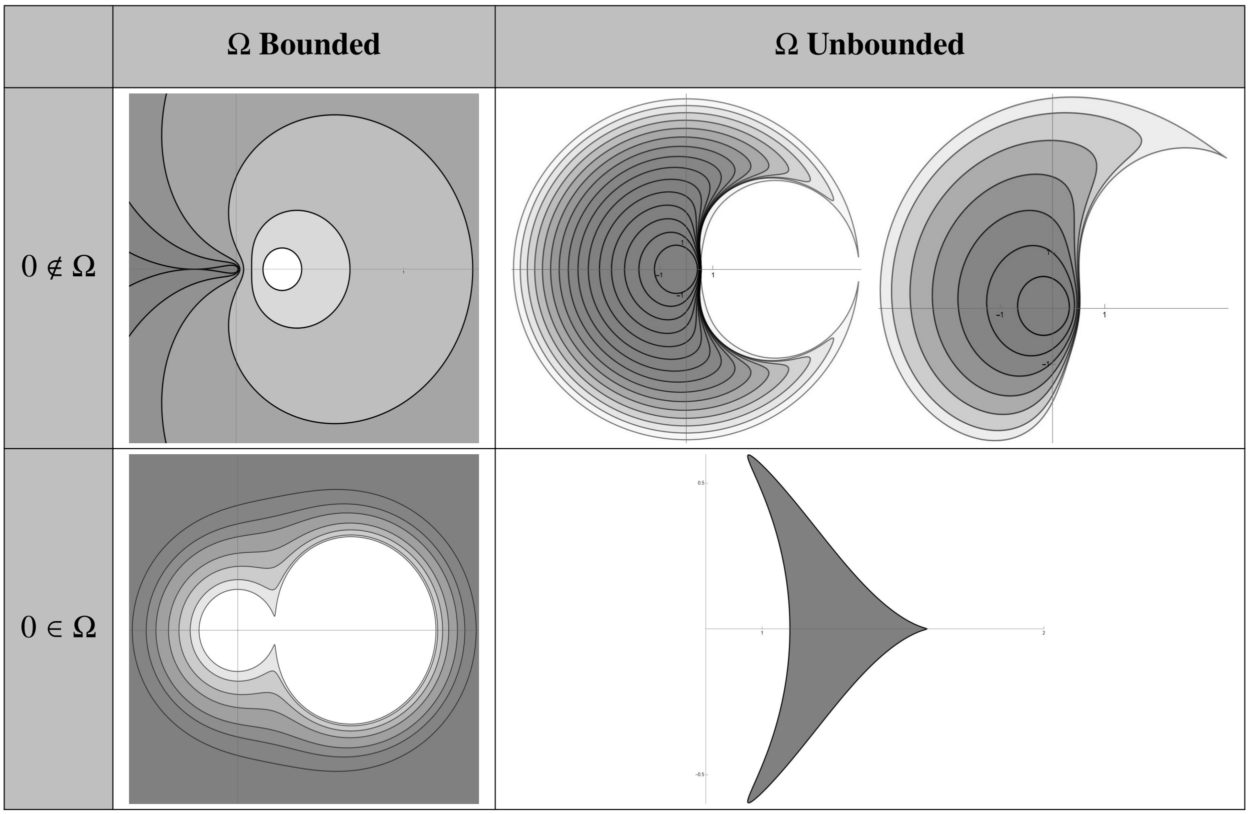

Figure 5.5: The families of (shaded complements of) one-point LQDs $\Omega \in \mathrm{QD}_0\left(\frac{\alpha}{w-w_0}\right)$ with

- $w_0 = .25$, $2 \leq \alpha \leq \pi^2$ (top left, §5.3);
- $w_0 = 1$, $\alpha = .7$, $q \in (-1.6, 2.5)$ (bottom left, §5.6);
- $w_0 = 1.9$, $\alpha = 1.5, 1.5 + .3i$ (top right, §5.6);
- $w_0 = 2$, $\alpha = -.15$, $c = .389$, $z_0 = -3.13$, $z_1 = 2.28$ (bottom right, §5.6).

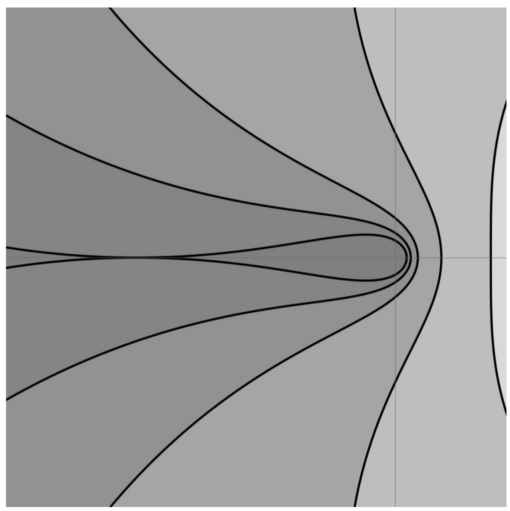

Figure 5.6: Zoom of double-point formation in Figure 5.5, $\Omega \in \mathrm{QD}_0\left(\frac{\alpha}{w-w_0}\right)$ with $w_0 = .25$ and $0 < \alpha \leq \pi^2$

Example: A Family of Two-Point LQDs

In this section, we consider a doubly symmetric family of simply connected LQDs with quadrature function $h(w) = \frac{\alpha}{w-1} + \frac{\alpha}{w+1}$ for $\alpha > 0$.

Theorem 5.6.4. *Fix $\alpha > 0$ and $q \in \mathbb{C}$. If $\Omega \in QD_0\left(\frac{\alpha}{w-1} + \frac{\alpha}{w+1}; q\right)$ is a simply connected singular LQD symmetric about the real and imaginary axes, then $\Omega = \varphi(\mathbb{D})$, where*

$$\varphi(z) = \frac{z}{z_+} e^{\lambda \frac{z^2 - z_+^2}{(zz_+)^2 - 1}}. \tag{5.18}$$

Moreover, $\varphi(z_+) = 1$, $\lambda = \frac{q + \ln(z_+^2)}{z_+^2 + z_+^{-2}}$, and $\alpha = \lambda^2 + \lambda \frac{z_+^4 - 1}{2z_+^2}$.

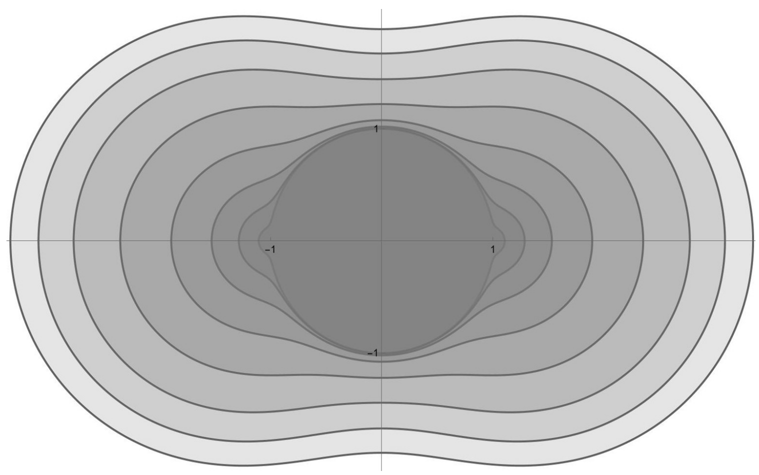

Figure 5.7: Family of LQDs in $\mathrm{QD}_0\left(\frac{\alpha}{w-1} + \frac{\alpha}{w+1}; q\right)$, for $q = 0$ and $0 < \alpha < 1$.

Proof of Theorem 5.6.4. Suppose that $\Omega \in \mathrm{QD}_0\left(\frac{\alpha}{w-1} + \frac{\alpha}{w+1}\right)$ for some $\alpha > 0$. If the Riemann map associated to Ω, $\varphi : \mathbb{D} \to \Omega$ takes 0 to 0, then $\varphi(z) = cze^{r^{\#}(z)}$, where $c = \varphi'(0) > 0$. Also choose $z_+, z_- \in \mathbb{D}$ such that $\varphi(z_\pm) = \pm 1$. By Theorem 5.4.2, $r(z) = \Phi_\varphi^{-1}\left(\frac{\alpha}{w-1} + \frac{\alpha}{w+1} - 2\alpha\right)(z)$. Applying the Faber transform formulae of Equation 2.11, we find

$$\begin{aligned} r(z) &= \alpha \Phi_\varphi^{-1}\left(\frac{1}{w-1}\right)(z) - \alpha \Phi_\varphi^{-1}\left(\frac{1}{w+1}\right)(z) \\ &= \frac{\alpha}{\varphi'(z_+)} \frac{1}{z - z_+} - \frac{\alpha}{\varphi'(z_-)} \frac{1}{z - z_-}. \end{aligned}$$

Assuming the domain is symmetric about the real and imaginary axes, we find that $\varphi(-z) = -\varphi(z)$, $\varphi'(-z) = \varphi'(z)$, and $z_- = -z_+ \in \mathbb{R}$, so this simplifies to $r(z) = \frac{2z_+\alpha}{\varphi'(z_+)} \frac{1}{z^2 - z_+^2}$, and we obtain

$$\varphi(z) = cze^{\frac{2z_+\alpha}{\varphi'(z_+)} \frac{z^2}{1 - z^2 z_+^2}}. \tag{5.19}$$

Using the relations for $\varphi(z_+)$ and $\varphi'(z_+)$, we find that

$$\varphi(z) = \frac{z}{z_+} e^{\frac{\ln(cz_+)}{z_+^2} \frac{z^2 - z_+^2}{(zz_+)^2 - 1}}.$$

We now compute q - that is, we know $\Omega \in \mathrm{QD}_0(h; q)$, and we would like to determine a relationship between the coefficients of φ and the charge q at 0. We compute the Schwarz function,

$$\begin{aligned}
\frac{\ln |w|^2}{w} &\doteq \frac{\ln(\varphi\varphi^{\#}) \circ \psi(w)}{w} \\
&= \frac{\frac{\ln(cz_+)}{z_+^2} \left(\frac{z^2 - z_+^2}{(zz_+)^2 - 1} + \frac{1 - (zz_+)^2}{z_+^2 - z^2} \right) \circ \psi(w) - \ln(z_+^2)}{w} \\
&= \frac{\ln\left(\frac{wz_+}{\psi(w)}\right) + \left(\frac{\ln(cz_+)}{z_+^2}\right)^2 \frac{1}{\ln\left(\frac{wz_+}{\psi(w)}\right)}}{w} - \frac{\ln(z_+^2)}{w}
\end{aligned}$$

By L'Hopital's rule,

$$\lim_{w \to 0} \frac{wz_+}{\psi(w)} = \frac{z_+}{\psi'(0)} = z_+ \varphi'(0) = cz_+.$$

So, expanding about $w = 0$, we obtain

$$\frac{\ln |w|^2}{w} \doteq \frac{\ln(cz_+)(z_+^{-4} + 1) - \ln(z_+^2)}{w} + O(w)$$

Hence, $q = \ln(cz_+)(z_+^{-4} + 1) - \ln(z_+^2)$. Setting $\lambda = \frac{q + \ln(z_+^2)}{z_+^2 + z_+^{-2}}$, we recover Equation 5.18. Finally, combining Equations 5.19 and 5.18, we obtain the equation for α in the theorem. $\square$

5.7 Aside: Transcendental Dynamics of The Schwarz Reflection

Let Ω be a quadrature domain with Schwarz function S and associated droplet $K = \Omega^c$. By removing the finitely many singular points of $\partial\Omega$ from K, we obtain the *fundamental tile* T. We can then consider an anti-holomorphic dynamical system under the action of the *Schwarz reflection* $\sigma := \overline{S} : \mathrm{Cl}(\Omega) \to \widehat{\mathbb{C}}$. This system admits a partition into a pair of sets invariant under the action of σ:

1. The *tiling set* T^∞, composed of the set of points which eventually escape to the fundamental tile under the action of σ;

2. the *non-escaping set* $A(\infty)$, those points which do not escape to the fundamental tile.

The tiling set has its name because it can be alternatively defined as the union of the fundamental tile T and each of its preimages under the iteration of σ. In certain cases, the dynamics of σ on T^∞ resemble that of a reflection group and the dynamics on $A(\infty)$ resemble that of an anti-polynomial on its filled Julia set.

Lee et al [21] demonstrate that when, T is the deltoid, this phenomenon manifests with σ as the "mating" of the anti-polynomial $f_0(z) = \bar{z}^2$ ($f_0 : \mathbb{D}^c \to \mathbb{D}^c$) and the reflection map $\rho : \mathrm{Cl}(\mathbb{D}) \setminus \mathring{\Pi} \to \mathrm{Cl}(\mathbb{D})$ of the ideal triangle group (Π is the hyperbolic triangle in $\mathbb{D}$). This means that the dynamics of f_0 and ρ can be glued together along $\partial\mathbb{D}$, resulting in a topological map η on $\widehat{\mathbb{C}}$, which is conformally conjugate to σ when $\widehat{\mathbb{C}}$ is equipped with the appropriate conformal structure ([21], Theorem 1.1).

Recall the exterior "teardrop" of Figure 5.3 (left), $\Omega \in \mathrm{QD}_0(1)$, which is the image of $\mathbb{D}_*$ under $\varphi(z) = ze^{z^{-1}}$. In this case, the associated transcendental Schwarz reflection $\sigma : \Omega \to \widehat{\mathbb{C}}$ is given by

$$\sigma(w) = \overline{S(w)} = \overline{\varphi^{\#} \circ \psi(w)} = \overline{w}^{-1} e^{-W(-\overline{w}^{-1}) - W(-\overline{w}^{-1})^{-1}}, \tag{5.20}$$

where W is the principal branch of the Lambert W function. We note the striking similarity between the non-escaping set of σ, and the filled Julia set of the anti-holomorphic exponential map $w \mapsto e^{\overline{w}-1}$ (Figure 5.8).

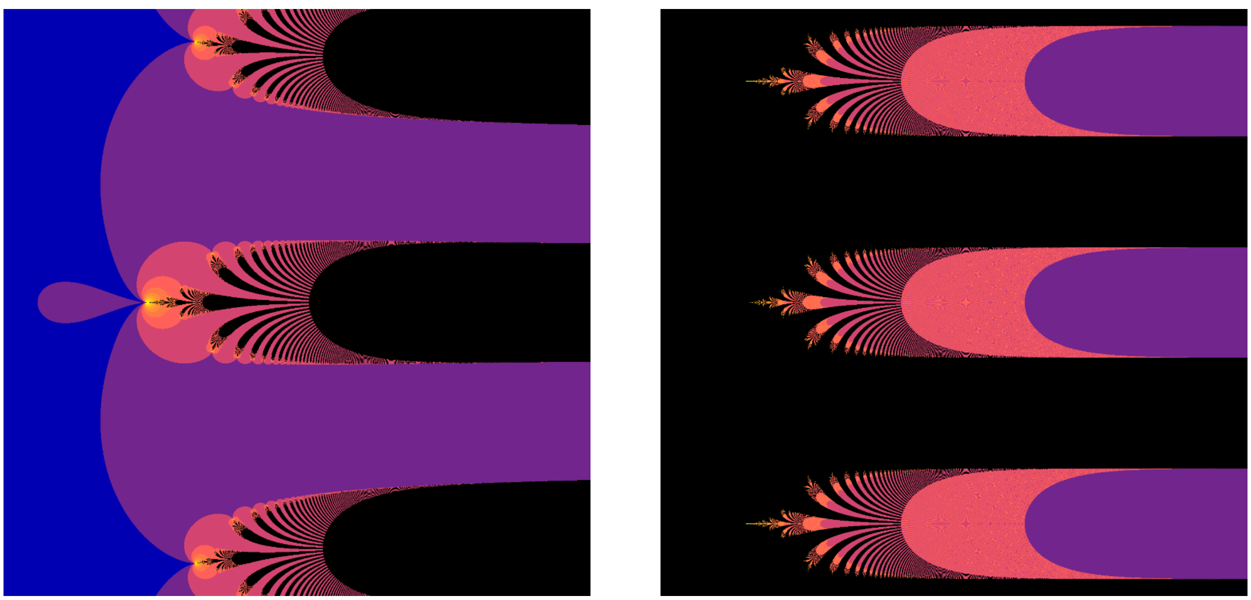

Figure 5.8: Tiling set of the Schwarz reflection σ associated to $\Omega \in \mathrm{QD}_0(1)$ (left) and filled Julia set of the anti-holomorphic exponential map $w \mapsto e^{\overline{w}-1}$ (right).

Chapter 6

ALGEBRAIC WEIGHTED QUADRATURE DOMAINS

6.1 Introduction

In §3 we introduced the notion of classical quadrature domains, and in §4-5 we studied two concrete families of weighted quadrature domains: those with the weight $\rho_a(w) = |w|^{2(a-1)}$ (PQDs) and those with the weight $\rho_0(w) = |w|^{-2}$ (LQDs). In each of these cases, the weight took the form $\rho = \frac{1}{4}\Delta F$ for a radially symmetric function F, and the resulting theory paralleled the classical case remarkably closely.

In this chapter, we broaden our scope to *algebraic* weights of the form $\rho_R = |R'|^2$, where R is a non-constant rational function. This class of weights is natural from several perspectives. First, the identity $|R'|^2 = \frac{1}{4}\Delta|R|^2$ ensures that the generalized Schwarz function machinery of Proposition 4.1.1 (strengthened by Proposition 6.2.1, to follow) applies. Second, *algebraic weighted quadrature domains* (AQDs) are intimately connected with classical QDs via the change of variables $w \mapsto R(w)$ (Proposition 6.3.1), which reduces many questions about AQDs to their classical counterparts. Third, the case $R(w) = w^n$ recovers the PQDs of Chapter 4.

The notion of an algebraic quadrature domain on a Riemann surface is due to Skinner [31]; we restrict our attention to $\widehat{\mathbb{C}}$ while relaxing some of Skinner's regularity assumptions.

6.2 Definition and Basic Properties

Fix a rectifiable domain $\Omega \subseteq \widehat{\mathbb{C}}$ and a non-constant rational function $R \in C^0(\partial\Omega)$. We define the test class

$$\mathcal{F}_R(\Omega) := \{f \in \mathcal{A}(\Omega) : f(w) = 0 \text{ at each pole of } R \text{ (including } \infty)\}.$$

Area integrals over $\mathcal{F}_R(\Omega)$ with respect to the weight $\rho_R = |R'|^2$ are understood in terms of their *R-principal value*:

$$\int_\Omega f|R'|^2 dA := \lim_{r\to\infty} \int_\Omega f|R'|^2 \mathbb{1}_{|R|<r} dA. \tag{6.1}$$

This is finite for $f \in \mathcal{F}_R(\Omega)$, as the following calculation shows. By Green's theorem,

$$\begin{aligned}\lim_{r\to\infty} \int_\Omega f|R'|^2 \mathbb{1}_{|R|<r} dA &= \oint_{\partial\Omega} fR\partial R - \lim_{r\to\infty} \oint_{\partial\{|R|<r\}\cap\Omega} fR\partial R \\ &= \oint_{\partial\Omega} fR\partial R - \lim_{r\to\infty} r^2 \oint_{\partial\{|R|<r\}\cap\Omega} f\frac{\partial R}{R} \\ &= \oint_{\partial\Omega} fR\partial R < \infty,\end{aligned}$$

where the last equality follows from the fact that f vanishes at each pole of R.

Definition 6.2.1 (Algebraic quadrature domain)**.** Let $\Omega \subset \widehat{\mathbb{C}}$ be a rectifiable domain and R a non-constant rational function with $R \in C^0(\partial\Omega)$. We say that Ω is an *algebraic quadrature domain with respect to* R if there exists a rational function h such that

$$\int_{\Omega} f|R'|^2 dA = \oint_{\partial\Omega} f(w)h(w)dw, \qquad \forall f \in \mathcal{F}_R(\Omega). \tag{6.2}$$

This is denoted by $\Omega \in \mathrm{QD}_R(h)$.[1]

We denote by $\Omega \in \mathrm{QD}_R$ that there exists a rational function h such that $\Omega \in \mathrm{QD}_R(h)$. As with the other classes of QDs considered in this manuscript, we will always require that $\Omega = \mathrm{Int}(\mathrm{Cl}(\Omega))$ because otherwise arbitrarily many AQDs with the same quadrature function could be constructed via deletion of subsets of measure zero.

Remark. When $R(w) = w$, we have $\rho_R = 1$ and $\mathcal{F}_R(\Omega) = \mathcal{A}_0(\Omega)$ (for unbounded Ω) or $\mathcal{A}(\Omega)$ (for bounded Ω), recovering the classical definition (Definitions 3.0.1 & 3.0.2). When $R(w) = \frac{w^a}{a}$ for $a \in \mathbb{Z}_+$, we recover the class of PQDs (Definition 4.2.1).

If $\Omega \in \mathrm{QD}_R(h)$ contains singularities of R, it is referred to as a *singular AQD*. We likewise refer to AQDs not containing singularities of R as *non-singular*.

The Generalized Coincidence Equation

The existence of a generalized coincidence equation for AQDs essentially follows from a strengthened version of Proposition 4.1.1.

Proposition 6.2.1. Let Ω be a bounded (resp. unbounded) domain, ρ a weight on Ω, $h \in \mathrm{Rat}(\Omega)$, and $\mathcal{F}$ a ρ-integrable analytic test class such that $\Omega \in \mathrm{QD}_\rho(h;\mathcal{F})$. Suppose $\rho = \frac{\partial^2 F}{\partial w \partial \overline{w}} = \frac{1}{4}\Delta F$ for some real-valued $F \in C^2(\partial\Omega)$ that is twice differentiable a.e. in Ω. If $m \in \mathcal{M}(\Omega)$ is such that $\frac{f}{m} \in \mathcal{F}$ for all $f \in \mathcal{A}(\Omega)$ (resp. $\mathcal{A}_0(\Omega)$), then there exists $G \in \mathcal{A}(\Omega)$ (resp. $\mathcal{A}_0(\Omega)$) such that

$$h(w) \doteq \frac{\partial F}{\partial w} + m(w)G(w). \tag{6.3}$$

Proof of Proposition 6.2.1. By assumption, if $f \in \mathcal{A}(\Omega)$ (resp. $\mathcal{A}_0(\Omega)$), then $\frac{f}{m} \in \mathcal{F}$, so applying the quadrature identity and Green's theorem yields

$$\oint_{\partial\Omega} \frac{f(w)}{m(w)} \left(h(w) - \frac{\partial F}{\partial w} \right) dw = \int_{\Omega} \frac{f}{m} \rho dA - \int_{\Omega} \frac{f}{m} \frac{\partial^2 F}{\partial w \partial \overline{w}} dA = 0.$$

Therefore, there exists $G \in \mathcal{A}(\Omega)$ (resp. $\mathcal{A}_0(\Omega)$) such that $h \doteq \frac{\partial F}{\partial w} + mG$. □

[1] This is equivalent to the statement that $\Omega \in \mathrm{QD}_{\rho_R}(h;\mathcal{F}_R(\Omega))$ and $\partial\Omega$ contains no singularities of R.

Now suppose $\Omega \in \mathrm{QD}_R(h)$ is a bounded domain and take $F = |R|^2$, $m(w) = \prod_{j=1}^m (w - p_j)^{-1}$, where the p_j are the finite poles of R in Ω. Then $\frac{f}{m} \in \mathcal{F}_R(\Omega)$ whenever $f \in \mathcal{A}(\Omega)$. When Ω is unbounded, taking $m(w) = (w - w_0)^m \prod_{j=1}^m (w - p_j)^{-1}$, where w_0 is any point in Ω_* yields $\frac{f}{m} \in \mathcal{F}_R(\Omega)$ whenever $f \in \mathcal{A}_0(\Omega)$. We conclude by Proposition 6.2.1:

Theorem 6.2.1. *If $\Omega \in QD_R(h)$ and $\{p_j\}_{j=1}^m$ are the finite poles of R in Ω, then there exists $G \in \mathcal{A}(\Omega)$ and constants $q_j \in \mathbb{C}$ such that*

$$R'(w)\overline{R(w)} \doteq h(w) + G(w) + \sum_{j=1}^m \frac{q_j}{w - p_j}, \tag{6.4}$$

We will now explicitly determine the analytic function G. Applying Cauchy's integral formula and substituting Equation 6.4, we obtain for each $w \in \Omega$,

$$G(w) = \oint_{\partial\Omega} \frac{G(\xi)}{\xi - w} d\xi = \oint_{\partial\Omega} \frac{R'(\xi)\overline{R(\xi)} - h(\xi) - \sum_{j=1}^m \frac{q_j}{\xi - p_j}}{\xi - w} d\xi = -\oint_{\partial\Omega_*} \frac{R'(\xi)\overline{R(\xi)}}{\xi - w} d\xi,$$

where the last step follows from the analyticity of $\frac{1}{\xi - w}\left(h(\xi) + \sum_{j=1}^m \frac{q_j}{\xi - p_j}\right)$ in Ω_*. Now let $\{p_j\}_{j=m+1}^n$ be the poles of R in Ω_*, fix positive constants $\{r_j\}_{j=m+1}^n$ and let U_j be the connected component of $\{|R(w)| > r_j\}$ containing p_j (likewise denote by U_∞ the component of $\{|R(w)| > r_\infty\}$ containing the pole at ∞, if any). Take the $r_j > 0$ sufficiently large that each U_j contains no other poles of R, the U_j are disjoint and do not intersect Ω. Let $U_- := U_\infty \sqcup \bigsqcup_{j=m+1}^n U_j$, so that

$$G(w) = -\oint_{\partial(\Omega_* \setminus U_-)} \frac{R'(\xi)\overline{R(\xi)}}{\xi - w} d\xi - \sum_{j=m+1}^n \oint_{\partial U_j} \frac{R'(\xi)\overline{R(\xi)}}{\xi - w} d\xi$$

Applying Green's theorem and the fact that $\xi \in \partial U_j \implies R(\xi)\overline{R(\xi)} = r_j^2$, we obtain

$$G(w) = \int_{\Omega_* \setminus U_-} \frac{|R'(\xi)|^2}{w - \xi} dA(\xi) - \sum_{j=m+1}^n r_j^2 \oint_{\partial U_j} \frac{R'(\xi)}{R(\xi)} \frac{d\xi}{\xi - w} = C_{\rho_R}^{\Omega_* \setminus U_-}(w) - \sum_{j=m+1}^n \frac{r_j^2}{w - p_j}.$$

We have proven the following useful corollary:

Corollary 6.2.1. If $\Omega \in \mathrm{QD}_R(h)$ and $\{p_j\}_{j=m+1}^n$ are the poles of R in Ω_*, then there exist positive constants $\{r_j\}_{j=m+1}^n$ such that the G in Theorem 6.2.1 satisfies

$$G(w) = C_{\rho_R}^{\Omega_* \setminus U_-}(w) - \sum_{j=m+1}^n \frac{r_j^2}{w - p_j}, \quad \forall w \in \Omega. \tag{6.5}$$

With the generalized coincidence equation established, our next goal is to characterize the conditions under which the quadrature function h associated to a given AQD is unique. In particular, we show that h is unique outright when Ω contains no poles of R, and unique modulo the addition of finitely many point charge terms $\frac{q_j}{w-p_j}$ when Ω contains poles p_k of Ω. Next, we recover an analogue of the Schwarz function for AQDs (Theorem 6.2.2): we introduce a *generalized Schwarz function* $S_{R,\Omega}$ whose boundary values satisfy $S_{R,\Omega} \doteq R'\overline{R}$, and show that Ω is an $R-$AQD if and only if such an $S_{R,\Omega}$ exists. We also prove a Sakai-type boundary regularity theorem (only finitely many singular boundary points, each a cusp or double point).

Essential Uniqueness of the Quadrature Function

A direct consequence of Theorem 6.2.1 is the essential uniqueness of the quadrature function associated to a given AQD. For non-singular AQDs, we obtain:

Corollary 6.2.2. If $\Omega \in \mathrm{QD}_R(h)$ is non-singular, then h is unique.

Proof. Suppose $\Omega \in \mathrm{QD}_R$ is non-singular and admits two quadrature functions h_1 and h_2 (that is, $\Omega \in \mathrm{QD}_R(h_1) \cap \mathrm{QD}_R(h_2)$). Then, h_1 and h_2 each satisfy Equation 6.4:

$$R'(w)\overline{R(w)} \doteq h_j(w) + G(w)$$

for the same G, as G is independent of h (it depends only on R and Ω). Subtracting these equations, we obtain $h_1 \doteq h_2$. It follows from the identity theorem that $h_1 = h_2$. □

On the other hand, if $\Omega \in \mathrm{QD}_R$ is singular (containing poles $\{p_j\}_{j=1}^m$ of R), then we only obtain uniqueness up to an m-parameter family of quadrature functions.

Corollary 6.2.3. If $\Omega \in \mathrm{QD}_R(h)$ is singular, then h is unique modulo the addition of m point charges $\frac{q_j}{w-p_j}$ at the poles $\{p_j\}_{j=1}^m$ of R in Ω. In particular, $\left\{\widetilde{h} : \Omega \in \mathrm{QD}_R(\widetilde{h})\right\} = \left\{h(w) + \sum_{j=1}^m \frac{q_j}{w-p_j} : q_j \in \mathbb{C}\right\}$.

Proof. Suppose $\Omega \in \mathrm{QD}_R$ is singular with two quadrature functions h_1 and h_2 (that is, $\Omega \in \mathrm{QD}_R(h_1) \cap \mathrm{QD}_R(h_2)$). Then, h_1 and h_2 each satisfy Equation 6.4:

$$R'(w)\overline{R(w)} \doteq h_l(w) + G(w) + \sum_{j=1}^m \frac{q_{l,j}}{w-p_j}$$

for the same G. Subtracting these equations and applying the identity theorem yields $h_1(w) - h_2(w) = \sum_{j=1}^m \frac{q_{2,j}-q_{1,j}}{w-p_j}$ for all $w \in \mathbb{C}$. In particular, there exist $q_j \in \mathbb{C}$ such that $h_2(w) = h_1(w) + \sum_{j=1}^m \frac{q_j}{w-p_j}$.

On the other hand, if $\Omega \in \mathrm{QD}_R(h)$, then $\Omega \in \mathrm{QD}_R\left(h(w) + \sum_{j=1}^m \frac{q_j}{w-p_j}\right)$ for all $q_j \in \mathbb{C}$ by inspection of the quadrature identity, using the fact that $f \in \mathcal{F}_R(\Omega) \Rightarrow f(p_j) = 0$. □

Considering Theorem 6.2.1 together with Corollary 6.2.3, we arrive at an apparent paradox: The former asserts that the q_j (in Equation 6.4) are uniquely associated to $\Omega \in \mathrm{QD}_R(h)$, whereas the latter asserts that $h(w) + \sum_{j=1}^m \frac{q_j}{w-p_j}$ is also a quadrature function for Ω for any collection of $q_j \in \mathbb{C}$. What is happening here? The root of the problem is that the q_j depend on both Ω and h. Hence we must instead consider the sum $h(w) + \sum_{j=1}^m \frac{q_j}{w-p_j}$ which is, indeed uniquely associated to Ω (this follows from essentially the same argument as in the proof of Corollary 6.2.3). Now consider the decomposition

$$h(w) + \sum_{j=1}^m \frac{q_j}{w-p_j} = \left(h(w) - \sum_{j=1}^m \frac{\operatorname{Res}_{p_j} h}{w-p_j} \right) + \sum_{j=1}^m \frac{q_j + \operatorname{Res}_{p_j} h}{w-p_j},$$

where the first term is the part which is residue-free at $\{p_j\}_{j=1}^m$, and the second captures those residues. It follows that the second term is, in fact, independent of h. This motivates the introduction of the following definition.

Definition 6.2.2. Fix constants $\{q_j\}_{j=1}^m$ and h rational such that $\operatorname{Res}_{p_j} h = 0$ for all $j \in \{1, \dots, m\}$. We denote by $\Omega \in \mathrm{QD}_R(h; q_1, \dots, q_m)$ that $\Omega \in \mathrm{QD}_R(h)$, and Equation 6.4 is satisfied for these q_j.

It is immediate that if $\Omega \in \mathrm{QD}_R(h; q_1, \dots, q_m)$, then the data $(h; q_1, \dots, q_m)$ is uniquely associated to Ω. From the perspective of potential theory, the q_j can be interpreted as the charges at the poles p_j of R in Ω.

The Generalized Schwarz Function and Boundary Regularity

Recall that quadrature domains may be equivalently characterized as those domains which admit a Schwarz function (Proposition 3.1.1). The generalized coincidence equation suggests that the correct analogue of the classical Schwarz function should have boundary values $R'\overline{R}$. We now show that this is indeed the right notion. In particular, we show that a domain Ω is an AQD if and only if it admits a *generalized Schwarz function*: A function $S_{R,\Omega} \in \mathcal{M}(\Omega)$ extending continuously to $\partial\Omega$ such that $S_{R,\Omega} \doteq R'\overline{R}$.

It is important to note that $S_{R,\Omega}$ *always has finitely many poles* - a fact which is clear when Ω is bounded. When Ω is unbounded, this follows from the fact that $S_{R,\Omega}$ is meromorphic at ∞, which means that the singularity at infinity (if any) is isolated.

Theorem 6.2.2. *If R is a rational function and Ω is a domain with $\partial\Omega$ containing no poles of R, then $\Omega \in QD_R$ if and only if it admits a generalized Schwarz function $S_{R,\Omega}$. In this case,*

$$S_{R,\Omega} = h(w) + G(w) + \sum_{j=1}^m \frac{q_j}{w-p_j}, \tag{6.6}$$

And $h(w) + \sum_{j=1}^{m} \dfrac{q_j}{w - p_j} = C_{\Omega_*}\left[S_{R,\Omega}\right](w)$. *$G$, h, the p_j and the q_j are as in Theorem 6.2.1.*

In this case, the generalized Schwarz function $S_{R,\Omega}$ is given by the RHS of Equation 6.4. In order to complete the proof of Theorem 6.2.2, we will need the following preliminary regularity result:

Lemma 6.2.1. If Ω admits a generalized Schwarz function $S_{R,\Omega}$ and $\partial\Omega$ contains roots $\{z_j\}_{j=1}^m$ of R' then for each $\epsilon > 0$, $(\partial\Omega) \setminus \cup_{j=1}^m \mathbb{D}_\epsilon(z_j)$ has finitely many singular points, each of which is either a cusp or a double point.

Proof of Theorem 6.2.1. Let $S_{R,\Omega}$ be a generalized Schwarz function for Ω, so $S_{R,\Omega} \doteq R'\overline{R}$. If $w_0 \in \partial\Omega$ is not a root of R' then, by the analytic inverse function theorem, there exists an open nbhd of w on which R has an analytic inverse. Hence, there exists an open nbhd of w_0 such that

$$S_{w_0}(w) := \overline{R^{-1}\left(\frac{\overline{S_{R,\Omega}(w)}}{\overline{R'(w)}}\right)} \doteq \overline{w}$$

is a local Schwarz function at w_0. In particular, if $\{z_j\}_{j=1}^m$ are the roots of R' on $\partial\Omega$, Ω admits a local Schwarz function at every point on $\partial\Omega$ except possibly at these z_j. We conclude by Sakai's regularity theorem (3.1.1) that for every $\epsilon > 0$, $(\partial\Omega) \setminus \cup_{j=1}^m \mathbb{D}_\epsilon(z_j)$ has finitely many singular points, each a cusp or a double point ($\partial\Omega$ has no degenerate points on account of the assumption that $\Omega = \mathrm{Int}(\mathrm{Cl}(\Omega))$). The finiteness of the singular set follows exactly as in the classical quadrature-domain case (§3.2 of [20]). □

To constrain the behavior at the singular points of R, we require an extra ingredient, that $R(\partial\Omega)$ is a subset of an algebraic curve:

Theorem 6.2.3. *If Ω admits a generalized Schwarz function $S_{R,\Omega}$, there exists a non-constant polynomial p, irreducible over $\mathbb{C}$, with real coefficients such that $p(x, y) = 0$ for all $x + iy \in R(\partial\Omega)$. (c.f. [1], Theorem 3). In particular, $\partial\Omega$ is a subset of an R−preimage of an algebraic curve.*

Proof of Theorem 6.2.3. By definition, $S_{R,\Omega}$ which is analytic in Ω except at finitely many poles such that $T := \frac{S_{R,\Omega}}{R'} \doteq \overline{R}$. Thus

$$f := \frac{1}{2}(R + T), \text{ and } g(w) := \frac{1}{2i}(R - T)$$

are analytic except at finitely many poles, and admit continuous real extensions to $\partial\Omega$, $f \doteq \mathrm{Re}(R)$, $g \doteq \mathrm{Im}(R)$. Now suppose Ω is bounded. We apply Theorem 2 of [1]:

Let $\Omega \subset \mathbb{C}$ be a bounded domain and suppose f and g are holomorphic in Ω except at finitely many poles. Further suppose that f and g admit real, continuous, extensions to $\partial\Omega$. Then there exists a *non-trivial polynomial $p(x, y)$, irreducible over $\mathbb{C}$, with real coefficients such that $p(f(z), g(z)) = 0$ on $\partial\Omega$.*

Therefore there exists a real non-constant, $\mathbb{C}$-irreducible polynomial p such that $p(f(w), g(w)) = 0$ on $\partial\Omega$. Thus, $p(\mathrm{Re}(R), \mathrm{Im}(R)) = 0$ on $\partial\Omega$. This is equivalent to $p(x, y) = 0$ for all $x+iy \in R(\partial\Omega)$. The argument is similar, but slightly more delicate in the unbounded case; see the proof of Theorem 3 in [1]. □

Theorem 6.2.3 straightforwardly implies that $\partial\Omega$ has finitely many singular points. We conclude:

Theorem 6.2.4. *If Ω admits a generalized Schwarz function $S_{R,\Omega}$, then $\partial\Omega$ has finitely many singular points, where each singular point which is not a root of R' is either a cusp or a double point.*

With this regularity result in hand, we are now prepared to prove Theorem 6.2.2.

Proof of Theorem 6.2.2. Suppose $\Omega \in \mathrm{QD}_R(h)$. Then by Theorem 6.2.1, Ω satisfies Equation 6.4. Set $S_{R,\Omega}$ equal to the RHS of the equation: $S_{R,\Omega}(w) := h(w) + G(w) + \sum_{j=1}^{m} \frac{q_j}{w-p_j}$. As $S_{R,\Omega} \in \mathcal{M}(\Omega)$ and extends continuously to the boundary, we find that $S_{R,\Omega} \doteq R'\overline{R}$ is a generalized Schwarz function for Ω.

On the other hand, suppose Ω admits a generalized Schwarz function $S_{R,\Omega}$. By Theorem 6.2.4, we know that $\partial\Omega$ is piecewise C^1, so Green's theorem applies. Hence, for each $f \in \mathcal{F}_R(\Omega)$,

$$\int_\Omega f|R'|^2 dA = \oint_{\partial\Omega} f(w)R'(w)\overline{R(w)}dw = \oint_{\partial\Omega} f(w)S_{R,\Omega}(w)dw.$$

Since $S_{R,\Omega}$ has finitely many poles in Ω, let h be the rational function obtained by summing the principal parts of $S_{R,\Omega}$ at those poles. Then

$$G = S_{R,\Omega} - h$$

is holomorphic in Ω and extends continuously to $\partial\Omega$. Hence,

$$\int_{\partial\Omega} f(w)G(w)dw = 0$$

by Cauchy's theorem. It follows that

$$\int_\Omega f|R'|^2 dA = \oint_{\partial\Omega} f(w)h(w)dw,$$

so $\Omega \in \mathrm{QD}_R(h)$. □

An immediate consequence of Theorem 6.2.2 and Theorem 6.2.4 is the following boundary regularity result for AQDs:

Theorem 6.2.5. *If $\Omega \in QD_R$ then $\partial\Omega$ has finitely many singular points, and each is a cusp, a double point, or located at a root of R'. In particular, $\partial\Omega$ is piecewise C^1.*

Proof. By Theorem 6.2.2, Ω has a generalized Schwarz function. The conclusion follows by Theorem 6.2.4. □

AQDs: An Electrostatic Perspective

As with LQDs, AQDs have a natural physical interpretation from the perspective of electrostatics. In particular, *a domain Ω is an AQD if and only if there is a finite configuration of point charges (and multipoles) on the interior of Ω whose electrostatic field in Ω^c is equal to that of an (appropriately renormalized) charge density ρ_R on Ω.* More formally,

Theorem 6.2.6. *Fix a rational function R and a domain Ω whose boundary contains no poles of R, and $\{p_j\}_{j=1}^m$ are the poles of R in Ω. Then $\Omega \in QD_R(h)$ if and only if there exist positive constants $\{r_j\}_{j=1}^m$ such that*

$$C_{\rho_R}^{\Omega\setminus U_+}(w) - \sum_{j=1}^{m} \frac{r_j^2}{w - p_j} = h(w) + \sum_{j=1}^{m} \frac{q_j}{w - p_j} \tag{6.7}$$

for all $w \in \Omega_*$.

Before proceeding with the proof of Theorem 6.2.6, we will need the following lemma which allows us to appropriately "renormalize" the weighted Cauchy transform:

Lemma 6.2.2. Let $\{p_j\}_{j=1}^n \subset \widehat{\mathbb{C}}$ be the finite poles of R, fix constants $r_j > 0$ and let U_j be the connected component of $\{|R(w)| > r_j\}$ containing p_j (likewise denote by U_∞ the component containing the pole at ∞, if any). Then,

$$C_{\rho_R}^{\widehat{\mathbb{C}}\setminus U}(w) = R'(w)\overline{R(w)} + \sum_{j=1}^{n} \frac{r_j^2}{w - p_j}, \quad \forall w \in U_*, \tag{6.8}$$

where $U = U_\infty \sqcup \bigsqcup_{j=1}^n U_j$.

Proof of Lemma 6.2.2. Take the r_j sufficiently large each U_j contains no other poles of R, the U_j are disjoint, and do not intersect $\partial\Omega$. By Green's theorem, for each $w \in U_*$,

$$\begin{aligned}
C_{\rho_R}^{U_*}(w) &= \int_{U_*} \frac{|R'(\xi)|^2 dA(\xi)}{w-\xi} \\
&= R'(w)\overline{R(w)} - \oint_{\partial U_*} \frac{R'(\xi)\overline{R(\xi)}d\xi}{\xi - w} \\
&= R'(w)\overline{R(w)} - \sum_{j=1}^{n} \oint_{\partial (U_j)_*} \frac{R'(\xi)\overline{R(\xi)}d\xi}{\xi - w} - \oint_{\partial (U_\infty)_*} \frac{R'(\xi)\overline{R(\xi)}d\xi}{\xi - w}
\end{aligned}$$

Applying the fact that $R\overline{R} \doteq r_j^2$ on ∂U_j, then the generalized argument principle, we obtain

$$\begin{aligned}
C_{\rho_R}^{U_*}(w) &= R'(w)\overline{R(w)} + \sum_{j=1}^{n} r_j^2 \oint_{\partial U_j} \frac{R'(\xi)}{R(\xi)} \frac{d\xi}{\xi - w} + r_\infty^2 \oint_{\partial U_\infty} \frac{R'(\xi)}{R(\xi)} \frac{d\xi}{\xi - w} \\
&= R'(w)\overline{R(w)} + \sum_{j} \frac{r_j^2}{w - p_j}.
\end{aligned}$$

Note that the U_∞ term drops out because $\frac{R'(\xi)}{R(\xi)} \frac{1}{\xi - w} \in O(\xi^{-2})$. $\square$

Proof of Theorem 6.2.6. For the forward direction, suppose that $\Omega \in \mathrm{QD}_R(h; q_1, \ldots, q_m)$. By Lemma 6.2.2, there exist positive constants $\{r_j\}_{j=1}^n$ such that Equation 6.8 holds. If $U_+ = U \cap \Omega$ and $U_- = U \cap \Omega_*$, then

$$\begin{aligned}
C_{\rho_R}^{\Omega\setminus U_+}(w) &= C_{\rho_R}^{\widehat{\mathbb{C}}\setminus U}(w) - C_{\rho_R}^{\Omega_*\setminus U_-}(w) \\
&= R'(w)\overline{R(w)} + \sum_{j=1}^{n} \frac{r_j^2}{w - p_j} - C_{\rho_R}^{\Omega_*\setminus U_-}(w)
\end{aligned}$$

By Theorem 6.2.1 and Corollary 6.2.1, for each $w \in \partial\Omega$,

$$C_{\rho_R}^{\Omega\setminus U_+}(w) = h(w) + G(w) + \sum_{j=1}^{m} \frac{q_j}{w - p_j} + \sum_{j=1}^{n} \frac{r_j^2}{w - p_j} - C_{\rho_R}^{\Omega_*\setminus U_-}(w),$$

where $G(w) = C_{\rho_R}^{\Omega_*\setminus U_-}(w) - \sum_{j=m+1}^{n} \frac{r_j^2}{w - p_j}$. Hence, for each $w \in \partial\Omega$,

$$C_{\rho_R}^{\Omega\setminus U_+}(w) - \sum_{j=1}^{m} \frac{r_j^2}{w - p_j} = h(w) + \sum_{j=1}^{m} \frac{q_j}{w - p_j}.$$

By the analyticity of $C_{\rho_R}^{\Omega\setminus U_+}$ in Ω_*, this equality extends to Ω_*, and we recover Equation 6.7.

For the reverse direction, suppose that there exist positive constants $\{r_j\}_{j=1}^m$ such that

$$C_{\rho_R}^{\Omega\setminus U_+}(w) - \sum_{j=1}^m \frac{r_j^2}{w-p_j} = h(w) + \sum_{j=1}^m \frac{q_j}{w-p_j}$$

for all $w \in \Omega_*$. Again by Lemma 6.2.2, for each $w \in (U_+)_* \supseteq \partial\Omega$,

$$C_{\rho_R}^{\Omega_*\setminus U_-}(w) = C_{\rho_R}^{\widehat{\mathbb{C}}\setminus U}(w) - C_{\rho_R}^{\Omega\setminus U_+}(w) = R'(w)\overline{R(w)} + \sum_{j=1}^n \frac{r_j^2}{w-p_j} - \left(h(w) + \sum_{j=1}^m \frac{q_j + r_j^2}{w-p_j}\right).$$

Hence, $R'(w)\overline{R(w)} \doteq C_{\rho_R}^{\Omega_*\setminus U_-}(w) - \sum_{j=1}^n \frac{r_j^2}{w-p_j} + C_{\rho_R}^{\Omega\setminus U_+}(w)$. If $f \in \mathcal{F}_R(\Omega)$ then Green's theorem implies

$$\begin{aligned}\int_\Omega f|R'|^2 dA &= \oint_{\partial\Omega} f(w)R'(w)\overline{R(w)}dw \\ &= \oint_{\partial\Omega} f(w)\left(C_{\rho_R}^{\Omega_*\setminus U_-}(w) + h(w) + \sum_{j=1}^m \frac{q_j}{w-p_j}\right)dw \\ &= \oint_{\partial\Omega} f(w)h(w)dw,\end{aligned}$$

where the last step follows from the fact that $f(w)C_{\rho_R}^{\Omega_*\setminus U_-}(w)$ and $f(w)\sum_{j=1}^m \frac{q_j}{w-p_j}$ are analytic in Ω. Hence, $\Omega \in \mathrm{QD}_R(h)$. □

6.3 The Change-of-Variables Principle

A powerful structural result for AQDs is that they are related to classical QDs by a change of variables. This principle, due to Skinner [31] (Lemma 5.2.6), reduces many questions about AQDs to their classical counterparts.

Proposition 6.3.1 (Change of variables for AQDs)**.** Let $\Omega, \Omega_R \subset \widehat{\mathbb{C}}$ be domains and suppose $R : \mathrm{Cl}(\Omega_R) \to \mathrm{Cl}(\Omega)$ is a bijective rational map with $R \in C^0(\partial\Omega_R)$. Then $\Omega \in \mathrm{QD}(h)$ if and only if $\Omega_R \in \mathrm{QD}_R(h_R)$, where h and h_R are related by

$$\begin{aligned} h_R(w) &= C_{(\Omega_R)_*}\left[h \circ R \cdot R'\right](w) = \oint_{\partial\Omega_*} \frac{h(\xi)d\xi}{R^{-1}(\xi) - w}, \\ h(w) &= C_{\Omega_*}\left[\left(\frac{h_R}{R'}\right) \circ R^{-1}\right](w) = \oint_{\partial(\Omega_R)_*} \frac{h_R(\xi)d\xi}{R(\xi) - w}. \end{aligned} \tag{6.9}$$

Proof. For the forward direction, suppose $\Omega \in \mathrm{QD}(h)$. If $f \in \mathcal{F}_R(\Omega_R)$, then $f \circ R^{-1} \in \mathcal{A}(\Omega)$ (resp. $\mathcal{A}_0(\Omega)$ when unbounded), so

$$\begin{aligned}\int_{\Omega_R} f|R'|^2 dA = \int_\Omega f \circ R^{-1} dA &= \oint_{\partial\Omega} f \circ R^{-1}(w)h(w)dw \\ &= \oint_{\partial\Omega_R} f(w)h \circ R(w)R'(w)dw = \oint_{\partial\Omega_R} f(w)C_{(\Omega_R)_*}\left[h \circ R \cdot R'\right](w)dw.\end{aligned}$$

The function $C_{(\Omega_R)_*}[h \circ R \cdot R']$ is rational because $h \circ R \cdot R' \in \mathcal{M}(\Omega_R) \cap C^0(\partial\Omega_R)$. The reverse direction is completely analogous. □

An important consequence is the following composition property: If P is another rational function then for domains on which $R \circ P$ is injective,

$$\mathrm{QD}_{R\circ P}(h_{R\circ P}) \xleftrightarrow{U \mapsto P(U)} \mathrm{QD}_R(h_R) \xleftrightarrow{U \mapsto R(U)} \mathrm{QD}(h).$$

Corollary 6.3.1 (PQD–negative-PQD duality)**.** Let $n \in \mathbb{Z}_+$ and suppose Ω is a domain with $0 \notin \partial\Omega$. Then $\Omega \in \mathrm{QD}_n(h_n)$ if and only if $\Omega^{-1} \in \mathrm{QD}_{w^{-n}}(h_{-n})$, where

$$h_{-n}(w) = \frac{c_h}{w} - \frac{h_n(w^{-1})}{w^2},$$

and $c_h = 0$ except possibly when $0 \in \Omega$ is bounded, in which case $c_h = \sum_p \operatorname*{Res}_p h_n$.

Corollary 6.3.2 (One-point AQDs are preimages of disks)**.** Suppose $\Omega \in \mathrm{QD}_R\left(\frac{\alpha}{w-w_0}\right)$ is a bounded, simply connected domain containing no singularities of R or zeros of R', with $\alpha > 0$ and $w_0 \in \Omega$. Then Ω is the connected component of $R^{-1}\left(\mathbb{D}_{\sqrt{\alpha}}(R(w_0))\right)$ containing w_0.

Proof of Corollary 6.3.2. Let $\varphi : \mathbb{D} \to \Omega$ be the Riemann map with $\varphi(0) = w_0$, $\varphi'(0) > 0$. Since the poles of R are absent from Ω, Equation 6.4 reduces to $R'\overline{R} \doteq h + G$ for some $G \in \mathcal{A}(\Omega)$. Precomposing with φ, we obtain on $\partial\mathbb{D}$:

$$\overline{R \circ \varphi} \doteq \left(\frac{h+G}{R'}\right) \circ \varphi.$$

Since $(R \circ \varphi)^\#$ is analytic in $\mathbb{D}_*$ with $(R\circ\varphi)^\#(\infty) = \overline{R(w_0)}$, the Cauchy decomposition gives

$$(R\circ\varphi)^\#(z) - \overline{R(w_0)} = C_{\mathbb{D}_*}\left[(R\circ\varphi)^\#\right](z) = C_{\mathbb{D}_*}\left[\overline{R\circ\varphi}\right](z) = C_{\mathbb{D}_*}\left[\frac{(h+G)\circ\varphi}{R'\circ\varphi}\right](z).$$

Now $G \circ \frac{\varphi}{R' \circ \varphi}$ is analytic in $\mathbb{D}$ (because $G \in \mathcal{A}(\Omega)$ and R' has no zeros in Ω by assumption), so $C_{\mathbb{D}_*}$ annihilates it. The remaining term is

$$C_{\mathbb{D}_*}\left[\frac{h\circ\varphi}{R'\circ\varphi}\right](z) = C_{\mathbb{D}_*}\left[\frac{\alpha}{(\varphi(z)-w_0)\,R'(\varphi(z))}\right] = \frac{\alpha}{R'(w_0)} C_{\mathbb{D}_*}\left[\frac{1}{\varphi(z)-w_0}\right] + C_{\mathbb{D}_*}[\text{reg.}](z),$$

where "reg." denotes $\frac{R'(\varphi(\xi))-R'(w_0)}{(\varphi(\xi)-w_0)R'(\varphi(\xi))} \cdot \frac{\alpha}{R'(w_0)}$, which is analytic in $\mathbb{D}$ (the apparent pole at $\xi = 0$ is removable because R' has no zeros in Ω). Thus $C_{\mathbb{D}_*}[\text{reg.}] = 0$, and the Faber transform formula

$$C_{\mathbb{D}_*}\left[\frac{1}{\varphi(z)-w_0}\right] = \Phi_\varphi^{-1}\left(\frac{1}{w-w_0}\right)(z) = \frac{\psi'(w_0)}{z-\psi(w_0)} = \frac{z^{-1}}{\varphi'(0)}$$

Yields

$$(R \circ \varphi)^{\#}(z) - \overline{R(w_0)} = \frac{\alpha}{\varphi'(0)R'(w_0)} z^{-1}.$$

Reflecting via # and using $\varphi'(0) = \frac{(R\circ\varphi)'(0)}{R'(w_0)}$, we obtain

$$R \circ \varphi(z) = R(w_0) + \frac{\alpha}{\overline{(R \circ \varphi)'(0)}} z.$$

Differentiating and evaluating at 0, we find $(R \circ \varphi)'(0) = \frac{\alpha}{\overline{(R\circ\varphi)'(0)}}$. Hence,

$$R \circ \varphi(z) = R(w_0) + \sqrt{\alpha} z.$$

In particular, Ω is a connected component of $R^{-1}(\mathbb{D}_{\sqrt{\alpha}}(R(w_0)))$. $\square$

This procedure involving the Faber transform can be generalized to construct Faber transform formulae similar to those for classical QDs (Theorem 3.2.2). Unfortunately, the inner/outer factorization (§2.4) approach used for PQDs (§4.3) and LQDs (§5.4) doesn't carry over to AQDs. This is essentially a consequence of the fact that the weight is no longer radially symmetric with respect to the origin. Before discussing this in more detail, we will briefly consider null AQDs.

Null AQDs

Theorem 6.3.1. *If $\Omega \in QD_R(0)$, then the ρ_R-weighted area of Ω is infinite. In particular, Ω must contain a pole of R or an open neighborhood of ∞ if $R(\infty) = \infty$.*

Proof. Since ρ_R is a.e. positive and Ω is open, the weighted area $A_{\rho_R}(\Omega) > 0$. If $A_{\rho_R}(\Omega) < \infty$, then $\mathbb{1}_\Omega \in L^1_a(\Omega; \rho_R) \subseteq \mathcal{F}$, and

$$0 = \int_\Omega \mathbb{1}_\Omega \rho_R dA = A_{\rho_R}(\Omega) > 0,$$

a contradiction. Hence $A_{\rho_R}(\Omega) = \infty$, which forces Ω to contain a pole of R (where $|R'|^2$ has a non-integrable singularity) or to extend to ∞ with $R(\infty) = \infty$. $\square$

Example 6.3.1 (Preimages of exterior disks)**.** If Ω is a connected component of $\{w \in \widehat{\mathbb{C}} : |R(w)| > a\}$ for some $a > 0$, then $|R(w)|^2 \doteq a^2$ on $\partial\Omega$. Hence, by Green's theorem and the generalized argument principle,

$$\begin{aligned}
\int_\Omega f|R'|^2 dA &= \lim_{r\to\infty} \int_{\Omega\cap\{|R|<r\}} f|R'|^2 dA \\
&= \oint_{\partial\Omega} f(w)R'(w)\overline{R(w)}dw - \lim_{r\to\infty} \oint_{\{|R(w)|=r\}\cap\Omega} f(w)R'(w)\overline{R(w)}dw \\
&= a^2 \oint_{\partial\Omega} f(w)\frac{R'(w)}{R(w)}dw - \lim_{r\to\infty} r^2 \oint_{\{|R(w)|=r\}\cap\Omega} f(w)\frac{R'(w)}{R(w)}dw
\end{aligned}$$

As f is zero at each pole of R, we find that $f\frac{R'}{R}$ has poles only at the zeros of R, and these poles are simple. As Ω is a connected component of $\{|R(w)| > a\}$, we know that Ω contains no zeros of R, so Ω contains no poles of $f\frac{R'}{R}$. Hence both integrals $= 0$ by Cauchy's theorem. We conclude that $\Omega \in \mathrm{QD}_R(0)$. This generalizes the well-known fact that exterior disks are null classical QDs.

While it is unclear whether the converse holds in general, it does hold for domains not containing roots of R'. In particular,

Corollary 6.3.3. If Ω is a domain containing no roots of R', then $\Omega \in \mathrm{QD}_R(0)$ if and only if Ω is a connected component of $\{w \in \widehat{\mathbb{C}} : |R(w)| > a\}$ for some $a > 0$.

Proof of Corollary 6.3.3. Suppose $\Omega \in \mathrm{QD}_R(0)$ and containins a subset $\{p_j\}_{j=1}^m$ of the poles of R. By Theorem 6.2.1, there exists $G \in \mathcal{A}(\Omega)$ such that $R'(w)\overline{R(w)} \doteq G(w) + \sum_{j=1}^m \frac{q_j}{w-p_j}$. Hence,

$$|R(w)|^2 \doteq \frac{R(w)}{R'(w)}\left(G(w) + \sum_{j=1}^m \frac{q_j}{w-p_j}\right)$$

As the logarithmic derivative of R, $\frac{R'(w)}{R(w)}$ has zeros at the zeros of R' which are not zeros of R, and has simple poles at the poles and zeros of R. Hence $\frac{R(w)}{R'(w)}$ has simple zeros at the poles and zeros of R, and poles at the zeros of R' which are not also zeros of R. In particular, the RHS has poles only at a subset of the zeros of R'.

If Ω contains no zeros of R', then the RHS is analytic in Ω. That is, there exists $\widehat{G} \in \mathcal{A}(\Omega)$ such that $|R(w)|^2 \doteq \widehat{G}$. Therefore $\mathrm{Im}(\widehat{G}) = 0$ so, by the maximum principle, $\widehat{G}$ is real-valued and analytic in Ω and hence constant on Ω. As $\mathrm{Re}(\widehat{G}) \geq 0$ on $\partial\Omega$, we conclude that $\widehat{G}$ is a non-negative real constant on Ω. In particular, $\exists a \geq 0$ such that $|R|^2 \doteq a^2$.

Moreover, by Theorem 6.3.1, Ω must either contain a pole of R, or an open neighborhood of ∞ if $R(\infty) = \infty$. Hence, Ω is a connected component of $\{w \in \widehat{\mathbb{C}} : |R(w)| > a\}$ for some $a \geq 0$. □

There are examples, however, of null AQDs containing roots of R' which are indeed a connected component of $\{w \in \widehat{\mathbb{C}} : |R(w)| > a\}$.

Example 6.3.2. Consider, for example the bounded simply connected domain $\Omega = \varphi(\mathbb{D})$, where

$$\varphi(z) = \frac{z}{\sqrt{\sqrt{2} - z^2}}.$$

Set $R(w) = \frac{1}{w^2-1}$, and note that $R \circ \varphi(z) = -\frac{z^2-\sqrt{2}}{2z^2-\sqrt{2}}$. Hence, for each $f \in \mathcal{F}_R(\Omega)$,

$$\begin{aligned}\int_\Omega f|R'|^2 dA &= \oint_{\partial\Omega} f(w)R'(w)\overline{R(w)}dw \\ &= \oint_{\partial\mathbb{D}} f \circ \varphi(z)(R \circ \varphi)'(z)\overline{R \circ \varphi(z)}dz \\ &= \oint_{\partial\mathbb{D}} f \circ \varphi(z)\frac{2\sqrt{2}z}{(2z^2-\sqrt{2})^2}\frac{z^{-2}-\sqrt{2}}{2z^{-2}-\sqrt{2}}dz \\ &= \frac{1}{\sqrt{2}}\oint_{\partial\mathbb{D}} f \circ \varphi(z)\frac{z}{(z^2-\sqrt{2})(z^2-\sqrt{2}^{-1})}dz.\end{aligned}$$

The only possible poles of the integrand in $\mathbb{D}$ are simple poles at $z = \pm 2^{-\frac{1}{4}}$, where $\varphi(\pm 2^{-\frac{1}{4}}) = \pm 1$. However, as $f \in \mathcal{F}_R(\Omega)$, $f(\pm 1) = 0$. Hence the integrand is analytic in $\mathbb{D}$, and we find that $\Omega \in \mathrm{QD}_R(0)$ and $0 \in \Omega$ is a root of R'.

Moreover, $\partial\Omega$ is indeed a level set of $|R|$, specifically, precomposing R with the Riemann map φ, we find that $\partial\Omega = \left\{w \in \mathbb{C} : |R|^2 = \frac{1}{2}\right\}$.

The above example shows that, while it is sufficient to assume in Corollary 6.3.3 that Ω contains no zeros of R', it is not strictly necessary.

6.4 The Faber Transform Method for Simply Connected AQDs

When $\Omega \in \mathrm{QD}_R(h)$ is simply connected, the Faber transform provides explicit formulae relating h, R, and the Riemann map φ. These formulae generalize Theorems 3.2.2, 3.2.3 and 4.3.1, 4.3.2.

Theorem 6.4.1 (Inverse problem for bounded AQDs)**.** *Suppose $\Omega \in QD_R(h)$ is a bounded simply connected domain with Riemann map $\varphi : \mathbb{D} \to \Omega$, where $\varphi(0) = w_0$ is not a pole of R. In this case,*

$$R \circ \varphi(z) = R(w_0) + \Phi_\varphi^{-1}\left(C_{\Omega_*}[R]\right)(z) - \Phi_\varphi^{-1}\left(C_{\Omega_*}[R]\right)(0) + r^\#(z), \tag{6.10}$$

where

$$r(z) = \Phi_\varphi^{-1}\left(C_{\Omega_*}\left[\frac{h(w) + G(w) + \sum_{j=1}^m \frac{q_j}{w-p_j}}{R'(w)}\right]\right)(z), \tag{6.11}$$

where G and q_j are as in Theorem 6.2.1.

Note that if Ω contains no roots of R', then Equation 6.11 simplifies to

$$r(z) = \Phi_\varphi^{-1}\left(\frac{h(w)}{R'(w)}\right)(z), \tag{6.12}$$

Proof of Theorem 6.4.1. Let $\{p_j\}_{j=1}^m$ be the poles of R in Ω with respective orders $\{n_j\}_{j=1}^m$, and $\varphi(z_j) = p_j$. Note that $R \circ \varphi$ is analytic in $\mathbb{D}$, except at the $\{z_j\}_{j=1}^m$. Hence, $(R \circ \varphi)^{\#}$ is analytic in $\mathbb{D}_*$ except at $\{\overline{z_j}^{-1}\}_{j=1}^m$. Thus there exist constants $\beta_{j,k} \in \mathbb{C}$ such that

$$C_{\mathbb{D}_*}\left[(R \circ \varphi)^{\#}\right](z) = (R \circ \varphi)^{\#}(z) - F(z), \quad z \in \mathbb{D}_*,$$

where $F(z) = \sum_{j=1}^m \sum_{k=1}^{n_j} \frac{\beta_{j,k}}{(z - \overline{z_j}^{-1})^k} + \overline{R(w_0)}$. From the coincidence equation (6.4), $\overline{R(w)} = \frac{S_{R,\Omega}(w)}{R'(w)}$ on $\partial\Omega$, where $S_{R,\Omega}(w) = h(w) + G(w) + \sum_{j=1}^m \frac{q_j}{w - p_j}$. Hence, on $\partial\mathbb{D}$, we have $(R \circ \varphi)^{\#} = \overline{R \circ \varphi} = \frac{S_{R,\Omega} \circ \varphi}{R' \circ \varphi}$, which implies

$$\begin{aligned}(R \circ \varphi)^{\#}(z) - F(z) &= C_{\mathbb{D}_*}\left[\frac{(h(w) + G(w) + \sum_{j=1}^m \frac{q_j}{w - p_j}) \circ \varphi}{R' \circ \varphi}\right](z)\\ &= \Phi_\varphi^{-1}\left(C_{\Omega_*}\left[\frac{h(w) + G(w) + \sum_{j=1}^m \frac{q_j}{w - p_j}}{R'(w)}\right]\right)(z)\end{aligned}$$

Reflecting about $\partial\mathbb{D}$ on both sides, we obtain

$$(R \circ \varphi)(z) = F^{\#}(z) + \Phi_\varphi^{-1}\left(C_{\Omega_*}\left[\frac{h(w) + G(w) + \sum_{j=1}^m \frac{q_j}{w - p_j}}{R'(w)}\right]\right)^{\#}(z)$$

We furthermore note, taking the analytic part of both sides that

$$\begin{aligned}C_{\mathbb{D}_*}\left[(R \circ \varphi)\right](z) &= C_{\mathbb{D}_*}\left[F^{\#}\right](z)\\ &= C_{\mathbb{D}_*}\left[F^{\#} - R(w_0) + \operatorname*{Res}_{\infty} F^{\#}\right](z) + C_{\mathbb{D}_*}\left[R(w_0) - \operatorname*{Res}_{\infty} F^{\#}\right](z)\\ &= F^{\#}(z) - R(w_0) + \operatorname*{Res}_{\infty} F^{\#}\end{aligned}$$

Applying the definition of the Faber transform, we get $C_{\mathbb{D}_*}\left[(R \circ \varphi)\right](z) = \Phi_\varphi^{-1}\left(C_{\Omega_*}[R]\right)(z)$, so

$$\Phi_\varphi^{-1}\left(C_{\Omega_*}[R]\right)(z) + R(w_0) - \operatorname*{Res}_{\infty} F^{\#} = F^{\#}(z)$$

In particular,

$$R \circ \varphi = R(w_0) - \operatorname*{Res}_{\infty} F^{\#} + \Phi_\varphi^{-1}\left(C_{\Omega_*}[R]\right) + \Phi_\varphi^{-1}\left(C_{\Omega_*}\left[\frac{h(w) + G(w) + \sum_{j=1}^m \frac{q_j}{w - p_j}}{R'(w)}\right]\right)^{\#}.$$

Evaluating at $z = 0$, we find

$$R(w_0) = R(w_0) - \operatorname*{Res}_{\infty} F^{\#} + \Phi_\varphi^{-1}\left(C_{\Omega_*}[R]\right)(0) + 0.$$

We conclude that

$$(R \circ \varphi)(z) = R(w_0) + \Phi_\varphi^{-1}\left(C_{\Omega_*}[R]\right)(z) - \Phi_\varphi^{-1}\left(C_{\Omega_*}[R]\right)(0) + r^{\#}(z),$$

where

$$r(z) = \Phi_\varphi^{-1}\left(C_{\Omega_*}\left[\frac{h(w) + G(w) + \sum_{j=1}^m \frac{q_j}{w-p_j}}{R'(w)}\right]\right)(z),$$

as desired. □

Applying analogous reasoning when $\varphi(0) = p_1$, a pole of R in Ω, we obtain the following variation of Theorem 6.4.1

Corollary 6.4.1. If $\Omega \in \mathrm{QD}_R(h)$ is bounded and simply connected with Riemann map $\varphi : \mathbb{D} \to \Omega$ with $\varphi(0) = w_0$, a pole of R then

$$R \circ \varphi(z) = \Phi_\varphi^{-1}\left(C_{\Omega_*}[R]\right)(z) + r^\#(z), \tag{6.13}$$

On the other hand, if Ω is unbounded, then we obtain

Theorem 6.4.2 (Inverse problem for unbounded AQDs)**.** *Suppose $\Omega \in QD_R(h)$ is an unbounded simply connected domain with Riemann map $\varphi : \mathbb{D}_* \to \Omega$, where $\varphi(\infty) = \infty$ and $\varphi'(\infty) = c > 0$. In this case,*

$$(R \circ \varphi)(z) = \Phi_\varphi^{-1}\left(C_{\Omega_*}[R]\right)(z) + r^\#(z) - r^\#(0), \tag{6.14}$$

where

$$r(z) = \Phi_\varphi^{-1}\left(C_{\Omega_*}\left[\frac{h(w) + G(w) + \sum_{j=1}^m \frac{q_j}{w-p_j}}{R'(w)}\right]\right)(z), \tag{6.15}$$

and G, q_j are as in Theorem 6.2.1.

Note that if Ω contains no roots of R', then the formula for r in Equation 6.15 simplifies to

$$r(z) = \Phi_\varphi^{-1}\left(\frac{h(w)}{R'(w)}\right)(z), \tag{6.16}$$

Proof of Theorem 6.4.2. Let $\{p_j\}_{j=1}^m$ be the poles of R in Ω with respective orders $\{n_j\}_{j=1}^m$, and $\varphi(z_j) = p_j$. Furthermore, let n_∞ be the order of the pole at ∞, if any. Note that $R \circ \varphi$ is analytic in $\mathbb{D}_*$, except at the $\{z_j\}_{j=1}^m$. Hence, $(R \circ \varphi)^\#$ is analytic in $\mathbb{D}$ except at $\{\overline{z_j}^{-1}\}_{j=1}^m$. Thus, there exist constants $\beta_{j,k} \in \mathbb{C}$ such that

$$C_{\mathbb{D}}\left[(R \circ \varphi)^\#\right](z) = (R \circ \varphi)^\#(z) - F(z), \quad z \in \mathbb{D},$$

where $F(z) = \sum_{k=1}^{n_\infty} \frac{\beta_{\infty,k}}{z^k} + \sum_{j=1}^m \sum_{k=1}^{n_j} \frac{\beta_{j,k}}{(z-\overline{z}_j^{-1})^k} \in \mathcal{A}_0(\mathbb{D}_*)$. By the coincidence equation (6.4), $\overline{R(w)} = \frac{S_{R,\Omega}(w)}{R'(w)}$ on $\partial\Omega$, where $S_{R,\Omega}(w) = h(w) + G(w) + \sum_{j=1}^m \frac{q_j}{w-p_j}$. Hence, $(R \circ \varphi)^\# \doteq \overline{R \circ \varphi} =$

$\frac{S_{R,\Omega}\circ\varphi}{R'\circ\varphi}$. Substituting, and applying the definition of the Faber transform, we obtain

$$(R\circ\varphi)^{\#}(z)-F(z)=C_{\mathbb{D}}\left[\frac{(h(w)+G(w)+\sum_{j=1}^{m}\frac{q_j}{w-p_j})\circ\varphi}{R'\circ\varphi}\right](z)$$
$$=\Phi_{\varphi}^{-1}\left(C_{\Omega_*}\left[\frac{h(w)+G(w)+\sum_{j=1}^{m}\frac{q_j}{w-p_j}}{R'(w)}\right]\right)(z)$$

Reflecting about $\partial\mathbb{D}$ yields

$$(R\circ\varphi)(z)=F^{\#}(z)+r^{\#}(z),$$

where

$$r(z)=\Phi_{\varphi}^{-1}\left(C_{\Omega_*}\left[\frac{h(w)+G(w)+\sum_{j=1}^{m}\frac{q_j}{w-p_j}}{R'(w)}\right]\right)(z)$$

Next, taking the analytic part of both sides yields

$$C_{\mathbb{D}}\left[(R\circ\varphi)\right](z)=C_{\mathbb{D}}\left[F^{\#}\right](z)+C_{\mathbb{D}}\left[r^{\#}\right](z)$$
$$=F^{\#}(z)+r^{\#}(0)$$

Applying the definition of the Faber transform, we get $C_{\mathbb{D}}\left[(R\circ\varphi)\right](z)=\Phi_{\varphi}^{-1}\left(C_{\Omega_*}\left[R\right]\right)(z)$, so

$$F^{\#}(z)=\Phi_{\varphi}^{-1}\left(C_{\Omega_*}\left[R\right]\right)(z)-r^{\#}(0).$$

In particular,

$$(R\circ\varphi)(z)=\Phi_{\varphi}^{-1}\left(C_{\Omega_*}\left[R\right]\right)(z)+r^{\#}(z)-r^{\#}(0),$$

as desired. $\square$

Theorem 6.4.3 (Direct problem for AQDs)**.** *If $\Omega\in QD_R(h)$ is simply connected (bounded or unbounded) with Riemann map φ, then $h(w)+\sum_{j=1}^{m}\frac{q_j}{w-p_j}=C_{\Omega_*}\left[R'\cdot(R\circ\varphi)^{\#}\circ\psi\right](w)$. In particular, when Ω is bounded,*

$$h(w)+\sum_{j=1}^{m}\frac{q_j}{w-p_j}=\Phi_{\varphi}\left(C_{\mathbb{D}_*}\left[R'\circ\varphi\cdot(R\circ\varphi)^{\#}\right]\right),\tag{6.17}$$

and when Ω is unbounded,

$$h(w)+\sum_{j=1}^{m}\frac{q_j}{w-p_j}=\Phi_{\varphi}\left(C_{\mathbb{D}}\left[R'\circ\varphi\cdot(R\circ\varphi)^{\#}\right]\right),\tag{6.18}$$

where the p_j and q_j are as in Theorem 6.2.1.

Proof. By Theorem 6.2.2, $h = C_{\Omega_*}\left[S_{R,\Omega}\right]$. On $\partial\Omega$, $S_{R,\Omega} = R'\overline{R} = R' \cdot (R \circ \varphi)^{\#} \circ \psi$, and the result follows from the definition of the Cauchy projection and Faber transform. □

From the above results, we conclude the following powerful generalization of Theorem 3.2.1.

Theorem 6.4.4. *Let R be rational and Ω a simply connected domain with Riemann map φ and $\partial\Omega$ containing no poles of R. Then $\Omega \in QD_R$ if and only if $R \circ \varphi$ extends to a rational function in $\widehat{\mathbb{C}}$.*

Proof of Theorem 6.4.4. For the forward direction, note that Theorems 6.4.1 and 6.4.2 tell us that we can write $R \circ \varphi$ as a linear combination of Faber transforms of Cauchy projections of meromorphic functions. The result then follows from the fact that the Cauchy projection of a meromorphic function is rational (Lemma 2.3.1), and the Faber transform of a rational function is rational (§2.3).

For the reverse direction, note that if $(R \circ \varphi) = F$ is rational, then $S_{R,\Omega} := R'F^{\#} \circ \psi$ is meromorphic in Ω, continuous up to the boundary, and satisfies the boundary condition

$$R'F^{\#} \circ \psi \doteq R'\overline{(R \circ \varphi) \circ \psi} = R'\overline{R}.$$

In particular, $S_{R,\Omega}$ is a generalized Schwarz function for Ω. We conclude by Theorem 6.2.2 that $\Omega \in \mathrm{QD}_R$. □

We will now proceed with several example applications of these Faber transform formulae to various weights ρ_R and quadrature functions h. As we shall see, the pole structure of the Riemann map is essentially determined by the poles and zeroes of R' in Ω, as well as the poles of h.

6.5 Examples

Example 6.5.1 (PQD of negative order)**.** Suppose $R(w) = w^{-n}$ and $0 \in \Omega \in \mathrm{QD}_R(\lambda w^{-2}; q)$ is a bounded, simply connected domain for some $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}_+$. By Theorem 6.4.1 and Corollary 6.4.1 there exist $G \in \mathcal{A}(\Omega)$ and $q \in \mathbb{C}$ such that the Riemann map φ (with $\varphi(0) = 0$) satisfies

$$\begin{aligned}
\varphi(z)^{-n} &= \Phi_\varphi^{-1}\left(w^{-n}\right) + \Phi_\varphi^{-1}\left(C_{\Omega_*}\left[\frac{\lambda w^{-2} + G(w) + \frac{q}{w}}{-nw^{-(n+1)}}\right]\right)^{\#}(z) \\
&= \sum_{k=1}^{n} \frac{\beta_k}{(z-\psi(0))^k} - \frac{1}{n}\Phi_\varphi^{-1}\left(C_{\Omega_*}\left[\lambda w^{n-1} + w^{n+1}G(w) + qw^n\right]\right)^{\#}(z) \\
&= \sum_{k=1}^{n} \beta_k z^{-k} - 0.
\end{aligned}$$

Applying Theorem 6.4.3, we find that for each $w \in \Omega_*$,

$$\begin{aligned}\lambda w^{-2} + qw^{-1} &= C_{\Omega_*}\left[(-nw^{-(n+1)}) \cdot (\varphi^{-n})^{\#} \circ \psi(w)\right] = -nC_{\Omega_*}\left[\frac{1}{w^{n+1}}\sum_{k=1}^{n}\overline{\beta_k}\psi^k(w)\right] \\ &= -n\sum_{k=1}^{n}\overline{\beta_k}\oint_{\partial\Omega_*}\frac{1}{\xi - w}\frac{\psi^k(\xi)}{\xi^{n+1}}d\xi\end{aligned}$$

Note that $\psi \in \mathcal{A}(\mathbb{D})$ and $\psi(0) = 0$. Hence, the kth integrand in the sum has a pole of order $n-k+1$ at 0. In particular, there exist constants $\gamma_{j,k}$ such that

$$\begin{aligned}\lambda w^{-2} + qw^{-1} &= n\sum_{k=1}^{n}\overline{\beta_k}\operatorname*{Res}_{\xi=0}\frac{1}{\xi - w}\frac{\psi^k(\xi)}{\xi^{n+1}} \\ &= n\sum_{k=1}^{n}\frac{\overline{\beta_k}}{(n-k)!}\left(\frac{1}{\xi - w}\frac{\psi^k(\xi)}{\xi^k}\right)^{(n-k)}(0) \\ &= n\sum_{k=1}^{n}\frac{\overline{\beta_k}}{(n-k)!}\sum_{j=1}^{n-k+1}\frac{\gamma_{j,k}}{w^j},\end{aligned}$$

where the leading $\gamma_{n-k+1,k}$ terms are all non-zero. Matching orders, we find that all of the $\beta_k = 0$ except for $k = n$ and $k = n-1$. That is,

$$\varphi^{-n}(z) = \beta_{n-1}z^{-(n-1)} + \beta_n z^{-n}.$$

Again applying Theorem 6.4.3, we find that for each $w \in \Omega_*$,

$$\begin{aligned}\lambda w^{-2} + qw^{-1} &= -nC_{\Omega_*}\left[\frac{\overline{\beta_{n-1}}\psi^{n-1}(w) + \overline{\beta_n}\psi^n(w)}{w^{n+1}}\right] \\ &= -\frac{n\overline{\beta_{n-1}}\beta_n^{\frac{n-1}{n}}}{w^2} - \frac{n|\beta_n|^2 + (n-1)|\beta_{n-1}|^2}{w}.\end{aligned}$$

In particular,

$$\begin{aligned}\lambda &= -n\overline{\beta_{n-1}}\beta_n^{\frac{n-1}{n}}, \\ q &= -n|\beta_n|^2 - (n-1)|\beta_{n-1}|^2.\end{aligned}$$

We find

$$\varphi^n(z) = \frac{z^n}{\beta_n}\left(1 - \overline{\lambda}\frac{\overline{\beta_n^{\frac{1}{n}}}}{n|\beta_n|^2}z\right)^{-1},$$

where $q = -n|\beta_n|^2 - \dfrac{n-1}{n^2}|\lambda|^2\,|\beta_n|^{-2\frac{n-1}{n}}$. Assuming that $\varphi'(0) = c > 0$, we find that $c^n = \beta_n^{-1}$. It follows that

$$\varphi(z) = cz\left(1 - \overline{\lambda}\frac{c^{\frac{2-n}{n}}}{n}z\right)^{-\frac{1}{n}},$$

where $(n-1)|\lambda|^2c^{2(2n-1)} + n^2qc^{2n} + n^3 = 0$. One can check that this is in agreement with the PQD-negative PQD duality asserted by Lemma 6.3.1. See Figure 6.1 for an example of such a domain with $n = 4$.

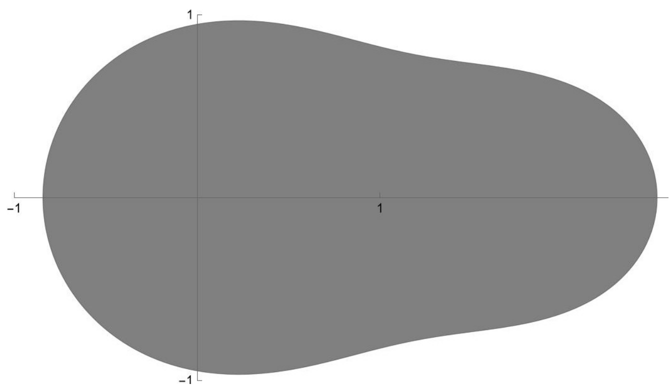

Figure 6.1: Example of negative one-point PQD with $n = 4$, $c = 1$, $\lambda = 3.9$, and $q = -1.148$.

A Doubly Singular Weight

Suppose $\Omega \in \mathrm{QD}_R(h)$ is bounded and simply connected with $R(w) = \dfrac{1}{w^2-1}$, and $\varphi : \mathbb{D} \to \Omega$ the associated Riemann map with $\varphi(0) = w_0$, not a pole of R, and $\varphi'(0) = c > 0$. By Theorem 6.4.1, φ is given by

$$\frac{1}{\varphi^2(z)-1} = \frac{1}{w_0^2-1} + \Phi_\varphi^{-1}\left(C_{\Omega_*}\left[\frac{1}{w^2-1}\right]\right)(z) - \Phi_\varphi^{-1}\left(C_{\Omega_*}\left[\frac{1}{w^2-1}\right]\right)(0) + r^\#(z),$$

where

$$r(z) = \Phi_\varphi^{-1}\left(C_{\Omega_*}\left[\frac{h(w) + G(w) + \frac{q_1}{w-1} + \frac{q_2}{w+1}}{-\frac{2w}{(w^2-1)^2}}\right]\right)(z),$$

and $q_1 = 0$ if $1 \notin \Omega$ and $q_2 = 0$ if $-1 \notin \Omega$. Computing the first Faber transform explicitly, we obtain

$$\begin{aligned}\Phi_\varphi^{-1}\left(C_{\Omega_*}\left[\frac{1}{w^2-1}\right]\right)(z) &= \frac{1}{2}\left(\Phi_\varphi^{-1}\left(C_{\Omega_*}\left[\frac{1}{w-1}\right]\right)(z) - \Phi_\varphi^{-1}\left(C_{\Omega_*}\left[\frac{1}{w+1}\right]\right)(z)\right)\\ &= \frac{1}{2}\left(\frac{\psi'(1)}{z-\psi(1)} - \frac{\psi'(-1)}{z-\psi(-1)}\right).\end{aligned}$$

So

$$\Phi_\varphi^{-1}\left(C_{\Omega_*}\left[\frac{1}{w^2-1}\right]\right)(z)-\Phi_\varphi^{-1}\left(C_{\Omega_*}\left[\frac{1}{w^2-1}\right]\right)(0)=\frac{z}{2}\left(\frac{z_1^{-1}\varphi'(z_1)^{-1}}{z-z_1}\mathbb{1}_{1\in\Omega}-\frac{z_{-1}^{-1}\varphi'(z_{-1})^{-1}}{z-z_{-1}}\mathbb{1}_{-1\in\Omega}\right),$$

where $\varphi(z_1)=1$ (when $1\in\Omega$) and $\varphi(z_{-1})=-1$ (when $-1\in\Omega$). Hence,

$$\frac{1}{\varphi^2(z)-1}=\frac{1}{w_0^2-1}+\frac{z}{2}\left(\frac{z_1^{-1}\varphi'(z_1)^{-1}}{z-z_1}\mathbb{1}_{1\in\Omega}-\frac{z_{-1}^{-1}\varphi'(z_{-1})^{-1}}{z-z_{-1}}\mathbb{1}_{-1\in\Omega}\right)+r^\#(z),$$

and

$$r(z)=-\frac{1}{2}\Phi_\varphi^{-1}\left(C_{\Omega_*}\left[(w^2-1)^2\frac{h(w)+G(w)}{w}\right]\right)(z)+\frac{q_1-q_2}{2}\Phi_\varphi^{-1}\left(C_{\Omega_*}\left[\frac{1}{w}\right]\right)(z)$$

Example 6.5.2 (Ω contains 0 but neither pole of R)**.** If $\Omega\in\mathrm{QD}_R(h)$ (as above), with $h(w)=\lambda w^{-1}$ and contains 0 but neither pole of R, then we may set $w_0=0$. Applying the formulae for the Faber transform provided in Equation 2.11, we obtain

$$\frac{1}{\varphi^2(z)-1}=-1+\left(\frac{\overline{\lambda\varphi''(0)}}{2c^3}-\frac{\overline{G(0)}}{2c}\right)z-\frac{\overline{\lambda}}{2c^2}z^2.$$

In particular, $R\circ\varphi$ is a polynomial of degree 2.

Example 6.5.3 (Ω contains both poles of R and 0)**.** If $\Omega\in\mathrm{QD}_R(h;q_1,q_2)$ (as above), with $h(w)=\lambda w^{-1}$ and contains both poles of R, as well as 0, then we may set $w_0=0$. Applying the formulae for the Faber transform provided in Equation 2.11, we obtain

$$\frac{1}{\varphi^2(z)-1}=-1+\frac{z}{2}\left(\frac{z_1^{-1}\varphi'(z_1)^{-1}}{z-z_1}-\frac{z_{-1}^{-1}\varphi'(z_{-1})^{-1}}{z-z_{-1}}\right)+r^\#(z),$$

and

$$r(z)=\left(\frac{\lambda\varphi''(0)}{2c^3}+\frac{q_1-q_2-G(0)}{2c}\right)\frac{1}{z}-\frac{\lambda}{2c^2}\frac{1}{z^2}$$

If we further assume for simplicity that $q_1=q_2,\lambda\in\mathbb{R}$ and Ω is symmetric about the real and imaginary axes. By Equations A.3 and A.4, this implies that $\varphi(-z)=-\varphi(z)$ and $\varphi(\overline{z})=\overline{\varphi(z)}$. In particular, this tells us that $z_{-1}=z_1\in\mathbb{R}$ and $\varphi'(z_{-1})=\varphi'(z_1)$, so

$$\frac{1}{\varphi^2(z)-1}=-1+\frac{z_1^{-1}}{\varphi'(z_1)}\frac{z^2}{z^2-z_1^2}+\left(\frac{\lambda\varphi''(0)}{2c^3}+\frac{q_1-q_2-G(0)}{2c}\right)z-\frac{\lambda}{2c^2}z^2,$$

Moreover, as $\varphi(-z)=-\varphi(z)$, this tells us that $\frac{1}{\varphi^2(-z)-1}=\frac{1}{\varphi^2(z)-1}$, so $\frac{\lambda\varphi''(0)}{2c^3}+\frac{q_1-q_2-G(0)}{2c}=0$. Thus the Riemann map formula simplifies to

$$\frac{1}{\varphi^2(z)-1}=-1+\frac{z_1^{-1}}{\varphi'(z_1)}\frac{z^2}{z^2-z_1^2}-\frac{\lambda}{2c^2}z^2.$$

Solving for φ, we obtain

$$\varphi(z) = cz\left(\frac{c^{-2}\left(\frac{\lambda}{2} - \frac{z_1^{-1}}{\varphi'(z_1)}\frac{c^2}{z^2-z_1^2}\right)}{c^2 + z^2\left(\frac{\lambda}{2} - \frac{z_1^{-1}}{\varphi'(z_1)}\frac{c^2}{z^2-z_1^2}\right)}\right)^{\frac{1}{2}}$$

Applying the constraint that $\varphi'(0) = c > 0$, we find that $\varphi'(z_1) = -\frac{2c^2}{z_1^3(\lambda-2c^4)}$. Substituting, we obtain

$$\varphi(z) = cz\left(\frac{c^{-2}\left(\lambda + z_1^2\frac{\lambda-2c^4}{z^2-z_1^2}\right)}{2c^2 + z^2\left(\lambda + z_1^2\frac{\lambda-2c^4}{z^2-z_1^2}\right)}\right)^{\frac{1}{2}}$$

Hence the problem of determining φ has been reduced to determining $c = \varphi'(0)$ and $z_1 = \varphi^{-1}(1)$. As a function of q_1, q_2, and λ.

Applying Theorem 6.4.3, we find (recall that we are assuming $q_1 = q_2$).

$$\begin{aligned}
\frac{\lambda}{w} + \frac{q_1}{w-1} + \frac{q_1}{w+1} &= C_{\Omega_*}\left[-\frac{2w}{(w^2-1)^2}\left(\frac{1}{\varphi^2-1}\right)^{\#} \circ \psi(w)\right] \\
&= -C_{\Omega_*}\left[\frac{2w}{(w^2-1)^2}\left(-1 - \frac{z_1^2(\lambda-2c^4)}{2c^2}\frac{z^2}{z^2-z_1^2} - \frac{\lambda}{2c^2}z^2\right)^{\#} \circ \psi(w)\right] \\
&= C_{\Omega_*}\left[\frac{2w}{(w^2-1)^2}\left(1 - \frac{z_1^2(\lambda-2c^4)}{2c^2}\frac{1}{\psi(w)^2 z_1^2 - 1} + \frac{\lambda}{2c^2}\frac{1}{\psi(w)^2}\right)\right]
\end{aligned}$$

Expanding in series using our formula for φ and computer algebra, we obtain

$$\frac{\lambda}{w} + \frac{q_1}{w-1} + \frac{q_1}{w+1} = \frac{\lambda}{w} + \frac{\alpha}{w-1} + \frac{\alpha}{w+1} + \frac{\beta}{(w-1)^2} + \frac{\beta}{(w+1)^2},$$

where $\alpha = -\frac{(2c^4-\lambda)(2c^4z_1^8+\lambda-2z_1^4\lambda)}{4c^4(z_1^4-1)}$ and $\beta = \frac{1}{2}\left(1 + \frac{c^2z_1^2 - \frac{\lambda}{2c^2z_1^2}}{z_1^4-1}\right)$. Hence, we obtain the relations $\beta = 0$ and $\alpha = q_1$. See Figure 6.2 for an example.

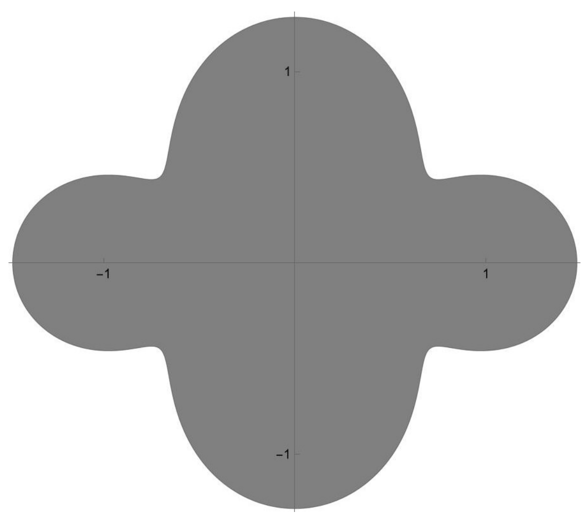

Figure 6.2: $\Omega \in \mathrm{QD}_R\left(\frac{\lambda}{w}; q_1, q_2\right)$, where $\lambda = \frac{2}{3}$ and $q_1 = q_2 = .1$. (Example 6.5.3)

Chapter 7

CONCLUSION

This thesis develops a theory of weighted quadrature domains parallel to the classical theory, and introduces the Faber transform as a tool for explicit construction and analysis of these domains. We now summarize the main contributions and discuss directions for future work.

Summary of contributions: A central point is the systematic use of the Faber transform for analysis of the inverse and direct problems for simply connected domains, enabling explicit constructions in many cases. Theorems 3.2.2 and 3.2.3 explicitly relate the quadrature function and the Riemann map via the Faber transform, reducing both problems to a problem of solving finitely many algebraic equations. This machinery is then applied to obtain a complete classification of one-point quadrature domains with complex charge (Theorem 3.3.1), extending the well-studied case of positive real charge to the full complex plane and identifying a parabolic existence region $|w_0|^2 + 2\mathrm{Re}(\alpha) > 2|\alpha|$, together with a characterization of the terminal cusp singularity.

In the power-weighted setting (Chapter 4), the key observation is that the correct object governing the simply connected theory is not the Riemann map itself, but its inner and outer factors: a domain is a PQD if and only if φ_{out}^{a} extends to a rational function (Theorem 4.3.3). This characterization leads to Faber transform formulae (Theorems 4.3.1, 4.3.2, 4.3.4, 4.3.6) that parallel the classical case while accommodating the new boundary phenomena - corners, cusps, and double points - which arise at the metric singularity.

The log-weighted theory, developed in Chapter 5, introduces novel complexities which arise from the non-integrable singularity of $\rho_0(w) = |w|^{-2}$ at the origin. The principal new phenomenon - that the quadrature function is determined only up to a point charge $\frac{q}{w}$ when $0 \in \Omega$ - necessitates reformulations of the coincidence equation, the Schwarz function, and the direct/inverse problems. Despite these complications, the simply connected theory admits a clean characterization: $\Omega \in \mathrm{QD}_0$ if and only if its outer part φ_{out} extends to the exponential of a rational function (Theorem 5.4.1). This exponential structure naturally reflects the role of the exponential map in connecting LQDs to classical quadrature domains (Theorem 5.1.1). Classification results for null LQDs (Theorem 5.3.1), one-point LQDs (Theorems 5.3.2 and 5.6.1), and monomial LQDs further demonstrate the tractability of the theory.

The algebraic framework of Chapter 6 contextualizes the preceding theories of PQDs and classical QDs in a larger setting. The change-of-variables principle (Proposition 6.3.1) shows that, on domains where R is injective, the map $w \mapsto R(w)$ establishes a correspondence $\mathrm{QD}_R \to \mathrm{QD}$

between algebraic and classical quadrature domains, reducing many questions about AQDs to known results. At the same time, the examples of Chapter 6 illustrate that AQDs exhibit richer behavior than their classical counterparts, including domains with multiple metric singularities and novel interactions between the analytic structure of R and the geometry of $\partial\Omega$.

Across all four classes of quadrature domains treated in this thesis, the Faber transform has served as the primary computational and structural tool. Its utility rests on three features: it is an isomorphism between function spaces on the disk and on the domain; it preserves rational functions; and it is explicitly computable via the calculus of residues. These properties make it particularly well-suited to problems in which the unknown data (the Riemann map or the quadrature function) is rational or admits a representation in terms of rational functions, as is the case for each class of quadrature domains considered here.

Open problems and future directions: Several natural questions remain.

First, the results of this thesis are largely restricted to the simply connected setting, where the inner/outer factorization of the Riemann map provides the key structural decomposition. Extending the Faber transform framework to multiply connected domains is a natural next step. The work of Crowdy and Marshall on Schottky-Klein prime functions [7] offers a promising approach to the multiply connected inverse problem for classical quadrature domains, and it would be of considerable interest to develop analogous tools in the weighted setting.

Second, while complete classifications of one-point QDs and null LQDs have been obtained, the classification of singular one-point and monomial LQDs remains incomplete. In particular, sharp results should be within reach even in the absence of rotational symmetry, and a full understanding of the interplay between the point charge parameter q and the geometry of the domain would deepen the theory.

Third, log-weighted quadrature domains are the first class of quadrature domains considered in this thesis for which the outer factor of the Riemann map is transcendental rather than rational. The exponential form of φ_{out} for LQDs naturally raises the question of whether other transcendental outer factors - for instance, those arising from other non-algebraic weights - admit tractable theories. More broadly, a systematic understanding of which weights ρ lead to tractable weighted quadrature domain theories, and what structural properties of ρ govern the analytic structure of the Riemann map, remains an open problem.

Fourth, the connections between quadrature domains and conformal dynamics have been explored primarily in the classical setting, where iteration of the Schwarz reflection associated to a quadrature domain yields a complex dynamical system and corresponding fractal tiling of the plane. In the log-weighted setting, the Schwarz reflection is transcendental, and preliminary observations (as noted in the companion paper with Makarov and in §5.7 of this thesis) suggest striking parallels between the non-escaping set of the transcendental Schwarz reflection and the filled Julia set of certain anti-holomorphic exponential maps. A rigorous analysis of these transcendental dynamical systems, along the lines of the work of Lee, Lyubich, Makarov, and Mukherjee [21] in the rational case, is a compelling direction for future research.

Finally, the duality between quadrature domains and droplets in logarithmic potential theory, which has been used throughout this thesis as both a tool and a source of motivation, suggests that progress on weighted quadrature domains may yield new results in the associated obstacle problems and equilibrium measure computations. In particular, the explicit Faber transform formulae developed here may provide new approaches to explicit construction of droplets for potentials of the form $Q(w) = \frac{|w|^{2a}}{a^2} - 2\mathrm{Re}(H(w))$ and $Q(w) = \frac{1}{2}\ln^2|w|^2 - 2\mathrm{Re}(H(w))$, where H' is rational.

It is the author's hope that the framework developed in this thesis - the systematic use of the Faber transform to mediate between quadrature data and conformal data in the simply connected setting - will prove useful in addressing these and related problems, and that the weighted theories introduced here will find further applications in potential theory, mathematical physics, and anti-holomorphic dynamics.

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Appendix A

PROOFS

A.1 Proof of Electrostatic Interpretation of LQDs

Proof of Theorem 5.1.6. We first consider the non-singular case ($0 \notin \Omega$):
For the forward direction, note that if $\Omega \in \mathrm{QD}_0(h)$ then the quadrature identity implies that for each $w \in \Omega_*$,

$$C_{\rho_0}^{\Omega}(w) = \int_{\Omega} \frac{|\xi|^{-2}}{w-\xi} dA(\xi) = \oint_{\partial\Omega} \frac{h(\xi)}{w-\xi} = h(w).$$

For the reverse direction, suppose that $C_{\rho_0}^{\Omega} = h$ on Ω_*. Note that this identity extends continuously to the boundary, as the poles of h are absent in an open neighborhood of $\partial\Omega$. Then, by Lemma 5.1.1, for each $w \in (\mathbb{D}_r)_* \supseteq \partial\Omega$,

$$\begin{aligned} C_{\rho_0}^{\Omega}(w) &= C_{\rho_0}^{\mathbb{C}\setminus\mathbb{D}_r}(w) - C_{\rho_0}^{\Omega_*\setminus\mathbb{D}_r}(w) \\ &= \frac{\ln|w|^2 - \ln|r|^2}{w} - C_{\rho_0}^{\Omega_*\setminus\mathbb{D}_r}(w). \end{aligned}$$

Substituting our formula $C_{\rho_0}^{\Omega}(w) = h(w)$, we find that for all $w \in \partial\Omega$,

$$\frac{\ln|w|^2}{w} = h(w) + \frac{\ln|r|^2}{w} + C_{\rho_0}^{\Omega_*\setminus\mathbb{D}_r}(w).$$

Hence, if $f \in L_a^1(\Omega;\rho_0)$ then Green's theorem implies

$$\begin{aligned} \int_{\Omega} \frac{f(w)}{|w|^2} dA(w) &= \oint_{\partial\Omega} f(w) \frac{\ln|w|^2}{w} dw \\ &= \oint_{\partial\Omega} f(w) \left(h(w) + \frac{\ln|r|^2}{w} + C_{\rho_0}^{\Omega_*\setminus\mathbb{D}_r}(w) \right) dw \\ &= \oint_{\partial\Omega} f(w)h(w)dw, \end{aligned}$$

where the last step follows from the fact that $f(w)C_{\rho_0}^{\Omega_*\setminus\mathbb{D}_r}(w)$ and $f(w)w^{-1}$ are analytic in Ω. Hence, $\Omega \in \mathrm{QD}_0(h)$.

We now consider the singular case ($0 \in \Omega$):
For the forward direction, suppose that $\Omega \in \mathrm{QD}_0(h;q)$. By Lemma 5.1.1

$$C_{\rho_0}^{\Omega\setminus\mathbb{D}_r}(w) = C_{\rho_0}^{\mathbb{C}\setminus\mathbb{D}_r}(w) - C_{\rho_0}^{\Omega_*}(w) = \frac{\ln|w|^2 - \ln|r|^2}{w} - C_{\rho_0}^{\Omega_*}(w).$$

By Theorem 5.1.3, for each $w \in \partial\Omega$,

$$C_{\rho_0}^{\Omega\setminus\mathbb{D}_r}(w) = \frac{\ln|w|^2 - \ln|r|^2}{w} - \left(\frac{\ln|w|^2}{w} - h(w) - \frac{q}{w}\right)$$
$$= h(w) + \frac{q - \ln|r|^2}{w}.$$

For the reverse direction, suppose that $C_{\rho_0}^{\Omega\setminus\mathbb{D}_r}(w) = h(w) + \dfrac{q - \ln|r|^2}{w}$ for all $w \in \Omega_*$. Again by Lemma 5.1.1, for each $w \in (\mathbb{D}_r)_* \supseteq \partial\Omega$,

$$C_{\rho_0}^{\Omega\setminus\mathbb{D}_r}(w) = C_{\rho_0}^{\mathbb{C}\setminus\mathbb{D}_r}(w) - C_{\rho_0}^{\Omega_*}(w) = \frac{\ln|w|^2 - \ln|r|^2}{w} - C_{\rho_0}^{\Omega_*}(w).$$

Hence, $\frac{\ln|w|^2}{w} \doteq h(w) + \dfrac{q}{w} + C_{\rho_0}^{\Omega_*}(w)$. If $f \in L_a^1(\Omega;\rho_0)$ then Green's theorem implies

$$\int_\Omega \frac{f(w)}{|w|^2} dA(w) = \oint_{\partial\Omega} f(w)\frac{\ln|w|^2}{w} dw$$
$$= \oint_{\partial\Omega} f(w)\left(h(w) + \frac{q}{w} + C_{\rho_0}^{\Omega_*}(w)\right) dw$$
$$= \oint_{\partial\Omega} f(w)h(w)dw,$$

where the last step follows from the fact that $f(w)C_{\rho_0}^{\Omega_*}(w)$ and $f(w)w^{-1}$ are analytic in Ω. Hence, $\Omega \in \mathrm{QD}_0(h)$. □

A.2 Inner/Outer Factorization Formulae

Classical Inner/Outer Factorization

Proof of Theorem 2.4.1.

We will begin by considering the **bounded** case: Let $\varphi : \mathbb{D} \to \Omega$ be the Riemann map associated to a simply connected bounded domain Ω with piecewise C^1 boundary such that $0 \notin \partial\Omega$. Then $\varphi \in H^\infty(\mathbb{D}) \cap C^0(\mathrm{Cl}(\mathbb{D}))$ by Caratheodory's theorem, and has no roots on $\partial\mathbb{D}$. Hence, by Lemma 2.4.1, we can write $\varphi = \varphi_{\mathrm{in}}\varphi_{\mathrm{out}}$, where φ_{in} is a Blaschke product consisting of the zeros of φ and φ_{out} is an outer function given by

$$\varphi_{\mathrm{out}}(z) = \exp\left(\frac{1}{2}\oint_{\partial\mathbb{D}} \frac{\xi + z}{\xi - z}\frac{\ln|\varphi(\xi)|^2}{\xi} d\xi\right).$$

Reflecting about the unit circle, we obtain for each $z \in \mathbb{D}_*$

$$\varphi_{\mathrm{out}}^{\#}(z) = \exp\left(\frac{1}{2}\oint_{\partial\mathbb{D}} \frac{\xi^{-1} + z^{-1}}{\xi^{-1} - z^{-1}}\frac{\ln|\varphi(\xi)|^2}{\xi^{-1}}\frac{d\xi}{\xi^2}\right) = \exp\left(\oint_{\partial\mathbb{D}_*} \frac{\ln|\varphi(\xi)|^2}{\xi - z} d\xi - \frac{1}{2}\oint_{\partial\mathbb{D}_*} \frac{\ln|\varphi(\xi)|^2}{\xi} d\xi\right)$$

The second integral is simply a constant depending on φ, and we recognize the first integral as the Cauchy projection onto $\mathbb{D}_*$. Hence, we obtain $\varphi^{\#}_{\text{out}}(z) = C\exp\left(\mathcal{C}_{\mathbb{D}_*}\left[\ln|\varphi(z)|^2\right]\right)$. As $\partial\Omega$ is piecewise C^1 and $\ln|w|^2 \in C^0(\partial\Omega)$, Equation 2.5 tells us that

$$\varphi_{\text{out}}(z) = \overline{C}\exp\left(\Phi_{\varphi}^{-1}\left(\mathcal{C}_{\Omega_*}\left[\ln|w|^2\right]\right)^{\#}(z)\right).$$

Moreover, by construction, if $0 \notin \Omega$, then $\varphi(z) \neq 0$, so $\varphi_{\text{in}} = 1$. On the other hand, if $0 \in \Omega$, then φ has a unique zero $z_0 \in \mathbb{D}$, in which case the Blaschke product collapses to a single factor $\varphi_{\text{in}} = b_{z_0}$.

We will now consider the **unbounded** case: Suppose $\varphi : \mathbb{D}_* \to \Omega$ is the Riemann map associated to a simply connected unbounded domain Ω with piecewise C^1 boundary such that $0 \notin \partial\Omega$, and set $f(z) := \frac{\varphi(z)}{z}$. Then $f \in H^\infty(\mathbb{D}_*) \cap C^0(\text{Cl}(\mathbb{D}_*))$ by Caratheodory's theorem, and has no roots in $\partial\mathbb{D}_*$. Hence, by Lemma 2.4.1, we can write $f = f_{\text{in}} f_{\text{out}}$, where f_{in} is a Blaschke product consisting of the zeros of φ and f_{out} is an outer function given by

$$f_{\text{out}}(z) = \exp\left(\frac{1}{2}\oint_{\partial\mathbb{D}_*}\frac{\xi+z}{\xi-z}\frac{\ln\left|\frac{\varphi(\xi)}{\xi}\right|^2}{\xi}d\xi\right) = \exp\left(\frac{1}{2}\oint_{\partial\mathbb{D}_*}\frac{\xi+z}{\xi-z}\frac{\ln|\varphi(\xi)|^2}{\xi}d\xi\right)$$

Reflecting about the unit circle, and taking $z \in \mathbb{D}$, we obtain

$$f^{\#}_{\text{out}}(z) = \exp\left(\frac{1}{2}\oint_{\partial\mathbb{D}_*}\frac{\xi^{-1}+z^{-1}}{\xi^{-1}-z^{-1}}\frac{\ln|\varphi(\xi)|^2}{\xi^{-1}}\frac{d\xi}{\xi^2}\right) = \exp\left(\oint_{\partial\mathbb{D}}\frac{\ln|\varphi(\xi)|^2}{\xi-z}d\xi - \frac{1}{2}\oint_{\partial\mathbb{D}}\frac{\ln|\varphi(\xi)|^2}{\xi}d\xi\right)$$

The second integral is simply a constant depending on φ, and we recognize the first integral as the Cauchy projection onto $\mathbb{D}$. Hence, we obtain $f^{\#}_{\text{out}}(z) = C\exp\left(\mathcal{C}_{\mathbb{D}}\left[\ln|\varphi(z)|^2\right]\right)$. As $\ln|w|^2 \in C^0(\partial\Omega)$, Equation 2.8 tells us that

$$f_{\text{out}}(z) = \overline{C}\exp\left(\Phi_{\varphi}^{-1}\left(\mathcal{C}_{\Omega_*}\left[\ln|w|^2\right]\right)^{\#}(z)\right).$$

By construction, if $0 \notin \Omega$, then $f(z) \neq 0$, so $f_{\text{in}} = 1$. On the other hand, if $0 \in \Omega$, then f has a unique zero $z_0 \in \mathbb{D}$, in which case the Blaschke product collapses to a single factor $f_{\text{in}} = b_{z_0}$. □

Power Factorization

To proceed with the proof of Theorem 2.15, we will need the Schwarz integral formula:

Lemma A.2.1. If $f \in \mathcal{A}(\mathbb{D})$, then for each $z \in \mathbb{D}$,

$$f(z) = \oint_{\partial\mathbb{D}}\frac{\xi+z}{\xi-z}\frac{\text{Re}(f(\xi))}{\xi}d\xi + i\text{Im}(f(0)) \tag{A.1}$$

On the other hand, if $f \in \mathcal{A}(\mathbb{D}_*)$ is bounded, then for each $z \in \mathbb{D}_*$,

$$f(z) = \oint_{\partial\mathbb{D}_*}\frac{\xi+z}{\xi-z}\frac{\text{Re}(f(\xi))}{\xi}d\xi + i\text{Im}(f(\infty)) \tag{A.2}$$

Equation A.1 is the standard Schwarz integral formula. Equation A.2 is obtained by applying Equation A.1 to the reflection $f^\#$ of f, then reflecting back.

Proof of Theorem 2.4.2.

Fix $a > 0$ and let Ω be a simply connected **bounded** domain with Riemann map $\varphi : \mathbb{D} \to \Omega$. Set $\varphi = \varphi_{\text{in}}\varphi_{\text{out}}$, where $\varphi_{\text{in}} = 1$ when $0 \notin \Omega$ and $\varphi_{\text{in}} = b_{z_0}$ when $0 \in \Omega$ and $\varphi(z_0) = 0$. Then φ_{out} is bounded and non-zero in $\mathbb{D}$, so there exists $f \in \mathcal{A}(\mathbb{D})$ such that $\varphi_{\text{out}}^a = f$. By Lemma A.2.1, for each $z \in \mathbb{D}_*$,

$$\begin{aligned} f^\#(z) &= \oint_{\partial\mathbb{D}_*} \frac{\xi + z}{\xi - z}\frac{\mathrm{Re}(f^\#(\xi))}{\xi} d\xi + i\mathrm{Im}(f(0)) \\ &= \oint_{\partial\mathbb{D}_*} \frac{2\mathrm{Re}(f^\#(\xi))}{\xi - z} d\xi - \oint_{\partial\mathbb{D}_*} \frac{\mathrm{Re}(f^\#(\xi))}{\xi} d\xi + i\mathrm{Im}(f(0)) \\ &= \oint_{\partial\mathbb{D}_*} \frac{f(\xi) + \overline{f(\xi)}}{\xi - z} d\xi + C, \end{aligned}$$

for some constant $C \in \mathbb{C}$ depending only on φ. Noting that $|\varphi|^{2a} = |\varphi_{\text{in}}|^{2a}|\varphi_{\text{out}}|^{2a} = f\overline{f}$, we find

$$\begin{aligned} f^\#(z) &= \oint_{\partial\mathbb{D}_*} \frac{f(\xi) + \frac{|\varphi(\xi)|^{2a}}{f(\xi)}}{\xi - z} d\xi + C \\ &= \oint_{\partial\mathbb{D}_*} \frac{\frac{|\varphi(\xi)|^{2a}}{f(\xi)}}{\xi - z} d\xi + C \\ &= \oint_{\partial\mathbb{D}_*} \frac{\left(\frac{|w|^{2a}}{f\circ\psi(w)}\right)\circ\varphi(\xi)}{\xi - z} d\xi + C \\ &= \oint_{\partial\mathbb{D}_*} \frac{\left(|w|^{2a}\left(\frac{\varphi_{\text{in}}\circ\psi(w)}{w}\right)^a\right)\circ\varphi(\xi)}{\xi - z} d\xi + C \\ &= \Phi_\varphi^{-1}\left(C_{\Omega_*}\left[|w|^{2a}\left(\frac{\varphi_{\text{in}}\circ\psi(w)}{w}\right)^a\right]\right)(z) + C \end{aligned}$$

Hence,

$$\varphi_{\text{out}}^a(z) = \Phi_\varphi^{-1}\left(C_{\Omega_*}\left[|w|^{2a}\left(\frac{\varphi_{\text{in}}\circ\psi(w)}{w}\right)^a\right]\right)^\#(z) + C$$

For some $C \in \mathbb{C}$ depending only on φ.

Fix $a > 0$ and let Ω be a simply connected **unbounded** domain with Riemann map $\varphi : \mathbb{D}_* \to \Omega$. Set $\varphi = \varphi_{\text{in}}\varphi_{\text{out}}$, where $\varphi_{\text{in}}(z) = z$ when $0 \notin \Omega$ and $\varphi_{\text{in}}(z) = zb_{z_0}(z)$ when $0 \in \Omega$ and $\varphi(z_0) = 0$.

Then φ_{out} is bounded and non-zero in $\mathbb{D}_*$, so there exists $f \in \mathcal{A}(\mathbb{D}_*)$ such that $\varphi_{\text{out}}^a = f$. By Lemma A.2.1, for each $z \in \mathbb{D}$,

$$\begin{aligned} f^\#(z) &= \oint_{\partial\mathbb{D}} \frac{\xi + z}{\xi - z} \frac{\text{Re}(f^\#(\xi))}{\xi} d\xi + i\text{Im}(f(\infty)) \\ &= \oint_{\partial\mathbb{D}} \frac{2\text{Re}(f^\#(\xi))}{\xi - z} d\xi - \oint_{\partial\mathbb{D}_*} \frac{\text{Re}(f^\#(\xi))}{\xi} d\xi + i\text{Im}(f(\infty)) \\ &= \oint_{\partial\mathbb{D}_*} \frac{f(\xi) + \overline{f(\xi)}}{\xi - z} d\xi + C, \end{aligned}$$

for some constant $C \in \mathbb{C}$ depending only on φ. Noting that $|\varphi|^{2a} = |\varphi_{\text{in}}|^{2a}|\varphi_{\text{out}}|^{2a} = f\overline{f}$, we find

$$\begin{aligned} f^\#(z) &= \oint_{\partial\mathbb{D}} \frac{f(\xi) + \frac{|\varphi(\xi)|^{2a}}{f(\xi)}}{\xi - z} d\xi + C \\ &= \oint_{\partial\mathbb{D}} \frac{\frac{|\varphi(\xi)|^{2a}}{f(\xi)}}{\xi - z} d\xi + C \\ &= \oint_{\partial\mathbb{D}} \frac{\left(\frac{|w|^{2a}}{f\circ\psi(w)}\right) \circ \varphi(\xi)}{\xi - z} d\xi + C \\ &= \oint_{\partial\mathbb{D}} \frac{\left(|w|^{2a} \left(\frac{\varphi_{\text{in}} \circ \psi(w)}{w}\right)^a\right) \circ \varphi(\xi)}{\xi - z} d\xi + C \\ &= \Phi_\varphi^{-1}\left(C_{\Omega_*}\left[|w|^{2a}\left(\frac{\varphi_{\text{in}} \circ \psi(w)}{w}\right)^a\right]\right)(z) + C \end{aligned}$$

Hence,

$$\varphi_{\text{out}}^a(z) = \Phi_\varphi^{-1}\left(C_{\Omega_*}\left[|w|^{2a}\left(\frac{\varphi_{\text{in}} \circ \psi(w)}{w}\right)^a\right]\right)^\# (z) + C$$

For some $C \in \mathbb{C}$ depending only on φ. □

A.3 Proofs of Univalence Criteria

Proof of Lemma 3.3.1. We shall prove the result only for bounded domains, as the proof in the unbounded case is analogous. Suppose that $f \in \mathcal{A}(\Omega)$ is continuous and injective on $\partial\Omega$, so $f(\partial\Omega)$ is a Jordan curve. Also let U be the region enclosed by $f(\partial\Omega)$. By the argument principle, the number of solutions to $f(z) = w_0$ for each $w_0 \in \widehat{\mathbb{C}}$ is given by

$$\oint_{\partial\Omega} \frac{f'(z)dw}{f(z) - w_0} = \oint_{f(\partial\Omega)} \frac{dw}{w - w_0} = \mathbb{1}_U(w_0).$$

That is, each $w_0 \in U$ is the image under f of a unique point in Ω and no point in U^c is in the image of f. In particular, f univalently maps Ω into the region enclosed by $f(\partial\Omega)$, so f is univalent in Ω. □

Proof of Proposition 4.4.3. The proposition is a straightforward consequence of the following result:

Let $a > 0$, $k, l \in \mathbb{Z}_+$ and, $\{z_j\}_{j=1}^k$, $\{w_j\}_{j=1}^l$ nonzero constants. If $\varphi : \mathbb{D}_* \to \widehat{\mathbb{C}}$ is given by

$$\varphi(z) = z\left(\frac{\left(1-\frac{z_1}{z}\right)\ldots\left(1-\frac{z_k}{z}\right)}{\left(1-\frac{w_1}{z}\right)\ldots\left(1-\frac{w_l}{z}\right)}\right)^{\frac{1}{a}}$$

then φ is univalent and $\varphi(\mathbb{D}_*)$ is starlike with respect to infinity if and only if each $|z_j|, |w_j| \leq 1$ and

$$a + Re\left(\sum_{j=1}^{k}\frac{z_j}{z-z_j} - \sum_{j=1}^{l}\frac{w_j}{z-w_j}\right) > 0, \quad \forall z \in \mathbb{D}_*.$$

It follows from Lemma 4.4.1 that an analytic function $f : \mathbb{D}_* \to \widehat{\mathbb{C}}$ is univalent and $\varphi(\mathbb{D}_*)$ is starlike with respect to infinity if and only if $\mathrm{Re}\left(z\frac{f'(z)}{f(z)}\right) > 0$ for all $z \in \mathbb{D}_*$. Moreover, by the chain rule, $\mathrm{Re}\left(z\frac{f'(z)}{f(z)}\right) > 0$ if and only if $\mathrm{Re}\left(z\frac{(f^a)'(z)}{f^a(z)}\right) > 0$. Thus, its sufficient to show that each $|z_j|, |w_j| \leq 1$ and

$$a + \mathrm{Re}\left(\sum_{j=1}^{k}\frac{z_j}{z-z_j} - \sum_{j=1}^{l}\frac{w_j}{z-w_j}\right) > 0, \quad \forall z \in \mathbb{D}_*$$

if and only if φ is analytic in $\mathbb{D}_*$ and $\mathrm{Re}\left(z\frac{(\varphi^a)'(z)}{\varphi^a(z)}\right) > 0$ in $\mathbb{D}_*$.

First, note that each $|z_j|, |w_j| \leq 1$ if and only if φ is analytic in $\mathbb{D}_*$. Second, we have that

$$\varphi^a(z) = z^{a-k+l}\frac{(z-z_1)\ldots(z-z_k)}{(z-w_1)\ldots(z-w_l)},$$

so

$$\mathrm{Re}\left(z\frac{(\varphi^a)'(z)}{\varphi^a(z)}\right) = \mathrm{Re}\left(z\frac{a-k+l}{z} + \sum_{j=1}^{k}\frac{z}{z-z_j} - \sum_{j=1}^{l}\frac{z}{z-w_j}\right) = a + \mathrm{Re}\left(\sum_{j=1}^{k}\frac{z_j}{z-z_j} - \sum_{j=1}^{l}\frac{w_j}{z-w_j}\right).$$

The desired result follows. □

A.4 Symmetry of the Domain Implies Symmetry of the Quadrature Function

Proof of Lemma 5.1.3. We will proceed with the proof of the lemma only in the singular case; however, the non-singular argument is entirely analogous. Suppose $0 \in \Omega$ is a $k-$fold rotationally symmetric disjoint union of LQDs, $\Omega = \bigsqcup_l \Omega_l$, $\Omega_l \in \mathrm{QD}_0(h_l)$ and we set $h = \sum_l h_l$. Also suppose Ω_0 is the connected component of Ω containing zero by Theorem 5.1.6, for each $w \in \Omega_*$,

$$h_0(w) = C_{\rho_0}^{\Omega_0\setminus\mathbb{D}_r}(w) - \frac{q - \ln|r|^2}{w} = \int_{\Omega_0\setminus\mathbb{D}_r}\frac{|\xi|^{-2}}{\xi - w}dA(w) - \frac{q-\ln|r|^2}{w},$$

$$h_{l>0}(w) = C_{\rho_0}^{\Omega_l}(w) = \int_{\Omega_l}\frac{|\xi|^{-2}}{\xi - w}dA(w).$$

Summing these expressions over l, we obtain

$$h(w) = C_{\rho_0}^{\Omega\setminus\mathbb{D}_r}(w) - \frac{q - \ln|r|^2}{w} = \int_{\Omega\setminus\mathbb{D}_r} \frac{|\xi|^{-2}}{\xi - w} dA(w) - \frac{q - \ln|r|^2}{w}.$$

Hence, applying the symmetry of Ω, we find that for each $j \in \mathbb{Z}$ and $w \in \Omega_*$

$$\begin{aligned} e^{2\pi i \frac{j}{k}} h(e^{2\pi i \frac{j}{k}} w) &= e^{2\pi i \frac{j}{k}} \int_{\Omega\setminus\mathbb{D}_r} \frac{|\xi|^{-2}}{\xi - e^{2\pi i \frac{j}{k}} w} dA(w) - e^{2\pi i \frac{j}{k}} \frac{q - \ln|r|^2}{e^{2\pi i \frac{j}{k}} w} \\ &= \int_{\Omega\setminus\mathbb{D}_r} \frac{|e^{2\pi i \frac{j}{k}} \xi|^{-2}}{e^{-2\pi i \frac{j}{k}} \xi - w} dA(w) - \frac{q - \ln|r|^2}{w} \\ &= \int_{\Omega\setminus\mathbb{D}_r} \frac{|\xi|^{-2}}{\xi - w} dA(w) - \frac{q - \ln|r|^2}{w} = h(w). \end{aligned}$$

By the identity theorem, we conclude that $e^{2\pi i \frac{j}{k}} h(e^{2\pi i \frac{j}{k}} w) = h(w)$ for all $j \in \mathbb{Z}$, and $w \in \mathbb{C}$. Now consider the function $f(w) = w^{-(k-1)} h(w)$, so that $f(e^{2\pi i \frac{j}{k}} w) = f(w)$.

If we write $f(w) = \frac{P_0(w)}{P_1(w)}$ for coprime polynomials P_0 and P_1, then we find that for each root z_0 of P_0 or P_1, every element of its orbit, $\{e^{2\pi i \frac{j}{k}} z_0\}$, is also a root, and of the same multiplicity. In particular if z_0 is a root of multiplicity n of P_i, then

$$P_i(w) = (w - z_0)^n (w - e^{2\pi i \frac{1}{k}} z_0)^n \ldots (w - e^{2\pi i \frac{k-1}{k}} z_0)^n \widetilde{P}_i(w) = (w^k - z_0^k)^n \widetilde{P}_i(w),$$

where $\widetilde{P}_i$ has no roots on the orbit of z_0. Iterating this procedure, we find that we can write P_0 and P_1 each as a product of factors of the form $(w^k - a^k)$. In particular, there exists a rational function g such that $f(w) = g(w^k)$. Hence, $h(w) = w^{k-1} g(w^k)$ for some rational function g. □

A.5 Symmetry of the Domain Implies Symmetry of Riemann Map

Suppose Ω is a bounded simply connected domain admitting a symmetry f. In particular, that f is a holomorphic automorphism of Ω and $f(\mathrm{Cl}(\Omega)) = \mathrm{Cl}(\Omega)$. Moreover, let $\varphi : \mathbb{D} \to \Omega$ be an associated Riemann map such that $\varphi(0) = w_0$ is a fixed point of f. Then $\widetilde{\varphi} := f \circ \varphi$ is also a Riemann map for Ω such that $\widetilde{\varphi}(0) = f(w_0) = w_0$. By the Riemann mapping theorem, this tells us that $\widetilde{\varphi}(z) = \varphi(\alpha z)$ for some $\alpha \in S^1$ (the two Riemann maps are identical up to a coordinate rotation). In particular, we have shown that

$$\varphi(\alpha z) = f \circ \varphi(z) \tag{A.3}$$

Differentiating with respect to z and evaluating at 0, we find that, in general, $\alpha = f'(w_0)$. We consider two specific examples:

- If f is a rotation about the origin by $\beta \in S^1$, then

$$\varphi(\beta z) = \beta \varphi(z).$$

- If f is a Mobius transformation, $f(w) = \frac{aw+b}{cw+d}$ fixing w_0 then

$$\varphi(\alpha z) = f \circ \varphi(z), \quad \alpha = \frac{ad - bc}{(cw_0 + d)^2}.$$

There is one last kind of symmetry to discuss: *Conjugate symmetry*. Suppose $\overline{\Omega} = \Omega$ is a bounded simply connected domain with Riemann map $\varphi : \mathbb{D} \to \Omega$ with $\varphi(0) = w_0 \in \mathbb{R}$ and $\varphi'(0) > 0$. Then $\widetilde{\varphi}(z) := \overline{\varphi(\overline{z})}$ is also a Riemann map for Ω such that $\widetilde{\varphi}(0) = w_0$ and $\widetilde{\varphi}'(0) > 0$. Hence, by the Riemann mapping theorem, $\widetilde{\varphi} = \varphi$, and we conclude

$$\varphi(z) = \overline{\varphi(\overline{z})}. \tag{A.4}$$